THE SPLITTING OF COHOMOLOGY OF p-GROUPS WITH RANK 2

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Abstract. Let p be an odd prime and BP the classifying space of a p-group P with $rank_p(P) = 2$. By using stable homotopy splitting of BP, we study the decomposition of $H^{even}(P;\mathbb{Z})/p$ and $CH^*(BP)/p$.

1. Introduction

Let P be a p-group and BP its classifying space. We study the stable splitting and splitting of cohomology

$$(*) \quad BP \cong X_1 \vee ... \vee X_i,$$

$$(**) \quad H^*(P) \cong H^*(X_1) \oplus ... \oplus H^*(X_i) \quad (for * > 0)$$

where X_i are irreducible spaces in the stable homotopy category. If we can get the splitting $(**)$ of cohomology, then we can study $H^*(G)_{(p)}$ for all groups G having Sylow p subgroups which are isomorphic to P.

When Mitchell and Priddy [Mi], [Mi-Pr] started this problem, the splitting $(*)$ was got by using the splitting $(**)$ of the cohomology of P. Since $H^*(P)$ are quite complicated for odd primes nonabelian p-groups, the examples were mostly given for $p = 2$.

But after the answer of the Segal conjecture by Carlsson, the splittings $(*)$ are given by only using modular representation theory by Nishida [Ni], Benson-Feshbach [Be-Fe] and Martino-Priddy [Ma-Pr]. In fact, their theorems say that such decomposition is decided only by structures of simple modules of the mod(p) double Burnside algebra $A(P,P)$. These theorems do not use splittings of cohomology, and results are given for all primes p.

In particular, Dietz and Dietz-Priddy [Di], [Di-Pr] gave the stable splitting $(*)$ for groups P with $rank_p(P) = 2$ for $p \geq 5$. However it was not used splittings $(**)$ of the cohomology $H^*(P)$, and the cohomologies $H^*(X_i)$ were not given there.

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In [Hi-Ya1,2], we give the cohomology of $H^*(X_i)$ (and hence $(**)$) for $P=p_+^{1+2}$ the extraspecial p group of order p^3 and exponent p. Their cohomology $H^*(X_i)$ are very complicated but have rich structures, in fact p_+^{1+2} is a p-Sylow subgroup of many interesting groups, e.g., $GL_3(\mathbb{F}_p)$ and many simple groups e.g. J_4 for $p=3$.

In this paper, we give the decomposition of

$$H^*(P)=H^*(P;\mathbb{Z})/(p,\sqrt{0}) \qquad (and\ \ H^{ev}(P)=H^{even}(P;\mathbb{Z})/p)$$

for other $rank_pP=2$ groups for odd primes p. It is important to compute the transfer map for the double Burnside algebra $A(P,P)$ action on $H^*(P;\mathbb{Z})$. In general, it is not a so easy problem to compute the transfer on $H^*(P;\mathbb{Z})$. However we can always compute it on $H^*(P)$ from Quillen's theorem.

In most cases, $H^*(X_i)$ are easily got but seemed not to have so rich structure as p_+^{1+2}, because they are not p-Sylow subgroups of so interesting groups. However, we hope that from our computations, it becomes more clear the relations among splittings of $H^*(P)$ of groups P with $rank_p(P)=2$.

In particular, we note that the irreducible components are most *fine* in those of $rank_p=2$ groups, namely, the cohomology $H^*(X_i(P))$ can be written by using the decomposition $H^*(X_k(p_+^{1+2}))$ (Lemma 8.1, Theorem 8.2, Theorem 8.5).

Theorem 1.1. *For $p\geq 5$, let P be a non-abelian p-group of $rank_pP=2$, which is not a metacyclic group. Let $X_i(P)$ be an irreducible component of BP. Then we can write*

$$H^*(X_i(P))\cong \oplus_{j\in J(i,P)}H^*(X_j(p_+^{1+2}))$$

for some index set $J(i,P)$.

This paper is organized as follows. In §2 we recall the definition and properties of the double Burnside algebra and the stable splitting. In §3, we note the relations between splittings when cohomology of groups are isomorphic. In $\S4-\S7$, we give the stable splitting of the cohomology of $H^*(P)$ for the elementary abelian group $\mathbb{Z}/p\times\mathbb{Z}/p$, metacyclic groups, $C(r)$ groups (such that $C(3)=p_+^{1+2}$), and $G(r',e)$ groups which appeared in the classification of $rank_p(P)=2$ groups for $p\geq 5$ respectively. However, some parts in $\S4,\S6$ are still given in [Hi-Ya1]. In §8, we study the relation among groups studied in $\S5-\S7$. In §9 we study the nilpotent ideal parts in $H^{ev}(P)=H^{even}(P;\mathbb{Z})/p$ for all groups in $\S4-\S7$. In §10, we note the relation to the Chow ring $CH^*(BP)/p$ and $H^{ev}(P)$, and note that the Chow group of the direct

summand X_i is represented by some motive of the classifying space BP.

2. THE DOUBLE BURNSIDE ALGEBRA AND STABLE SPLITTING

Let us fix an odd prime p and $k = \mathbb{F}_p$. For finite groups G_1, G_2, let $A_{\mathbb{Z}}(G_1, G_2)$ be the double Burnside group defined by the Grothendieck group generated by (G_1, G_2)-bisets. Each element Φ in $A_{\mathbb{Z}}(G_1, G_2)$ is generated by elements $[Q, \phi] = (G_1 \times_{(Q,\phi)} G_2)$ for some subgroup $Q \leq G_1$ and a homomorphism $\phi : Q \to G_2$. In this paper, we use the notation

$$[Q, \phi] = \Phi : G_1 \geq Q \xrightarrow{\phi} G_2.$$

For each element $\Phi = [Q, \phi] \in A_{\mathbb{Z}}(G_1, G_2)$, we can define a map from $H^*(G_2; k)$ to $H^*(G_1; k)$ by

$$x \cdot \Phi = x \cdot [Q, \phi] = Tr_Q^{G_1} \phi^*(x) \quad for\ x \in H^*(G_2; k).$$

When $G_1 = G_2$, the group $A_{\mathbb{Z}}(G_1, G_2)$ has the natural ring structure, and call it the (integral) double Burnside algebra. In particular, for a finite group G, we have an $A_{\mathbb{Z}}(G, G)$-module structure on $H^*(G; k)$ (and $H^*(G; \mathbb{Z})/p$).

Recall Quillen's theorem such that the restriction map

$$H^*(G; k) \to \lim_{V} H^*(V; k)$$

is an F-isomorphism (i.e. the kernel and cokernel are nilpotent) where V ranges elementary abelian p-subgroups of G. We easily see ([Hi-Ya1])

Lemma 2.1. *Let $\sqrt{0}$ be the nilpotent ideal in $H^*(G; k)$ (or $H^*(G; \mathbb{Z})/p$). Then $\sqrt{0}$ itself is an $A_{\mathbb{Z}}(G, G)$-module.*

In this paper we first (in $\S 4 - \S 8$) consider the cohomology modulo nilpotent elements, since it is not so complicated from the above Quillen's theorem. Hence we write it simply

$$H^*(G) = H^*(G; \mathbb{Z})/(p, \sqrt{0}).$$

However we also compute $H^{even}(G; \mathbb{Z})/p$ in §9 below.

By the preceding lemma, we see that $H^*(G)$ has the $A_{\mathbb{Z}}(G, G)$-module structure. (Here note that $A_{\mathbb{Z}}(G, G)$ acts on unstable cohomology.) For ease of notations and arguments, when there is an $A_{\mathbb{Z}}(G, G)$-filtration $F_1 \subset \ldots \subset F_n \cong H^*(P)$ such that

$$grH^*(P) = \oplus F_{i+1}/F_i \cong \oplus m_i M_i \quad for\ * > 0$$

with simple $A_{\mathbb{Z}}(G,G)$-modules M_i, we simply write

$$H^*(G) \leftrightarrow \oplus m_i M_i.$$

Throughout this paper, we assume that degree $* > 0$ (or we consider $H^*(-)$ as the reduced theory $\tilde{H}^*(-)$). (We consider $H^*(G)$ as an element in $K_0(Mod(A_{\mathbb{Z}}(G,G)))$.) In this paper, $H^*(G) \cong A$ for an graded ring A means an graded module isomorphism otherwise stated, while (in most cases) it is induced from the ring isomorphism $grH^*(G) \cong A$ for some filtrations of $H^*(G)$.

Let $BG = BG_p$ be the p-completion of the classifying space of G. Recall that $\{BG, BG\}_p$ is the (p-completed) group generated by stable homotopy self maps. It is well known from the Segal conjecture (Carlsson's theorem) that this group is isomorphic to the double Burnside group $A_{\mathbb{Z}}(G_1,G_2)^{\wedge}$ completed by the augmentation ideal.

Since the transfer is represented as a stable homotopy map Tr, an element $\Phi = [Q,\phi] \in A(G_1,G_2)$ is represented as a map $\Phi \in \{BG_1, BG_2\}_p$

$$\Phi : BG_1 \stackrel{Tr}{\to} BQ \stackrel{B\phi}{\to} BG_2.$$

(Of course, the action for $x \in H^*(G_2)$ is given by $Tr_Q^{G_1}\phi^*(x)$ as stated.)

Let us write

$$A(G_1,G_2) = A_{\mathbb{Z}}(G_1,G_2) \otimes k \quad (k = \mathbb{Z}/p).$$

Hereafter we consider the cases $G_i = P$; p-groups. Given a primitive idempotents decomposition of the unity of $A(P,P)$

$$1 = e_1 + ... + e_n,$$

we have an indecomposable stable splitting

$$BP \cong X_1 \vee ... \vee X_n \quad with\ e_iBP = X_i.$$

In this paper, an isomorphism $X \cong Y$ for spaces means that it is a stable homotopy equivalence.

Recall that

$$M_i = A(P,P)e_i/(rad(A(P,P)e_i)$$

is a simple $A(P,P)$-module where $rad(-)$ is the Jacobson radical. By Wedderburn's theorem, the above decomposition is also written as

$$BP \cong \vee_j(\vee_k X_{jk}) = \vee_j m_j X_{j1} \quad where\ m_j = dim(M_j)$$

for $A(P,P)e_{jk}/rad(A(P,P)e_{jk}) \cong M_j$. Therefore the stable splitting of BP is completely determined by the idempotent decomposition of the unity in the double Burnside algebra $A(P,P)$.

For a simple $A(P,P)$-module M, define a stable summand $X(M)$ by

$$e_M = \sum_{M_i \cong M} e_i, \quad X(M) = \vee_{M_{jk}\cong M} X_{jk} = e_M BP.$$

Here $X(M)$ is only defined in the stable homotopy category. (So strictly, the cohomology ring $H^*(X(M))$ is not defined.) However we can define $H^*(X(M))$ as a graded submodule of the cohomology ring $H^*(P)$ by

$$H^*(X(M)) = H^*(P) \cdot e_M \quad (= e_M^* H^*(P) \ stabely)$$

where we think $e_M \in A(P,P)$ (rather than $e_M \in \{BP, BP\}$).

Lemma 2.2. *Given a simple $A(P,P)$-module M, the cohomology $grH^*(X(M))$ is isomorphic to a sum of M, i.e., (for $* > 0$)*

$$H^*(X(M)) \leftrightarrow \oplus_{i=1} M[k_i], \qquad 0 \le k_1 \le ... \le k_s \le ...$$

where $[k_s]$ is the operation ascending degree k_s.

Proof. Let M' be a simple $A(P,P)$-module such that $M' \ncong M$. Then

$$e_{M'} X(M_i) = e_{M'} e_M BP = pt.$$

Hence $e_{M'} : H^*(P) \to H^*(P)$ restricts $e_{M'} | H^*(X(M)) = 0$ (we assumed $* > 0$). This means that $H^*(X(M))$ does not contain M' as a summand. □

From Benson-Feshbach [Be-Fe] and Martino-Priddy [Ma-Pr], it is known that each simple $A(P,P)$-module is written as

$$S(P,Q,V) \quad for\ Q \le P,\ and\ V : simple\ k[Out(Q)] - module.$$

(In fact $S(P,Q,V)$ is simple or zero.) Moreover, we have (see [Be-Fe]) an isomorphism

$$S(P,Q,V) \cong V \cdot A(P,Q)/J(Q,V)$$

for some $A(P,P)$-submodule $J(Q,V)$ of $V \cdot A(P,Q)$.

Thus we have the main theorem of stable splitting of BP.

Theorem 2.3. *(Benson-Feshbach [Be-Fe], Martino-Priddy [Ma-Pr]) There are indecomposable stable spaces $X_{S(P,Q,V)}$ for $S(P,Q,V) \neq 0$ such that*

$$BP \cong \vee X(S(P,Q,V)) \cong \vee (dimS(P,Q,V)) X_{S(P,Q,V)}.$$

The direct summands $X_{S(P,P,V)}$ are called dominant summands ([Ni], [Ma-Pr]). Let $X_{S=S(P,Q,V)}$ be a non-dominant summand for a proper subgroup Q. Then it is known ([Ni], [Ma-Pr]) that the corresponding

idempotent $e_S \in A(P,P)$ is generated by elements $P > Q \xrightarrow{\phi} P$ and $P \to Q \to P$. Hence when there is no non-trivial map $P \to Q$, we see

$$H^*(X_S) \cong e_S H^*(BP) \subset Tr_Q^P H^*(Q),$$

that is, $x \in H^*(X_S)$ if and only if $\sum Tr_Q^P \phi^*(x) = x$.

3. relation among groups P

Let R be a subring of $A(P,P)$. For a simple R-module S_R, we can define the idempotent e_{S_R} and the stable space $Y_{S_R} = e_{S_R}BP$ which decomposes BP, while it is (in general) not irreducible. In particular, we take the group algebra of the outer automorphism group $Out(P)$ as the ring R.

Lemma 3.1. *For each $Out(P)$-simple module S_{R_i} with dimension n_i, let us write by*

$$BP = n_1 Y_1 \vee ... \vee n_s Y_s \quad \textit{where } Y_i = e_{S_{R_i}} BP$$

the decomposition for idempotents in $Out(P)$. Then each Y_i decomposes

$$Y_i = X_{i_1} \vee ... \vee X_{i_m} \quad \textit{for } X_{ij} = e_{S_{ij}} BP$$

where $e_{S_{ij}}$ are idempotents in $A(P,P)$.

When P,P' are different p groups, the stable homotopy types of BP, BP' are different [Ni]. However there are many cases with $H^*(P) \cong H^*(P')$ (However note that it seems not so often that $H^*(P;\mathbb{Z}) \cong H^*(P';\mathbb{Z})$ even if we do not assume the map $P \to P'$ which induces the isomorphism on cohomology.) The following corollary is immediate from the above lemma.

Corollary 3.2. *Let P, P' are p-groups with $i_H : H^*(P) \cong H^*(P')$. Assume that there is a ring map $i_A : A(P,P) \to A(P',P')$ such that $i_H(\Phi(x)) = i_A(\Phi) i_H(x)$ for all $\Phi \in A(P,P)$ and $x \in H^*(P)$. Then for each splitting summand X_i in BP, there are splitting summands X'_{ij} of BP' such that*

$$i_H^*(X_i) = H^*(X'_{i_1}) \oplus ... \oplus H^*(X'_{i_s}).$$

Proof. We get the result from

$$i_H H^*(X_i) = i_H(e_i H^*(P)) = i_A(e_i) i_H H^*(P) = i_A(e_i) H^*(P').$$

□

Proposition 3.3. *Let $f : BP \to BG$ be a map such that $f^* : H^*(G) \to H^*(P)$ is injective. Let $BP = \vee X_i(P)$ and $BG = \vee X_j(G)$ be the irreducible decompositions. For each $X_j(G)$, there are $i_1, .., i_s$ such that*

$$H^*(X_j(G)) \cong f^*H^*(G) \cap (H^*(X_{i_1}(P) \vee ... \vee X_{i_s}(P)).$$

Proof. We note each $f(X_i(P))$ is contained some $X_j(G)$ otherwise $X_i(P))$ should be decomposed. Let $f(X_{i_k}(P)) \subset X_j(G)$. Then we have a map

$$f^* : H^*(X_j(G)) \to H^*(X_{i_1}(P) \vee .... \vee X_{i_s}(P)).$$

Since f^* is injective, we have the proposition. □

If G has Sylow p-group isomorphic to P, then of course the above proposition holds. Moreover we consider the cases that $P \subset G$ and G is also a p-group satisfying the above proposition in §5 below.

4. $A = \mathbb{Z}/p \times \mathbb{Z}/p$ FOR $p \geq 3$

In this section, we recall the decomposition of the cohomology of $\mathbb{Z}/p \times \mathbb{Z}/p$, which is still given §5 in [Hi-Ya1]. However the result is not so trivial, and we write it briefly. (The results used in the other sections are only Lemma 4.1 and Theorem 4.4.)

We recall the cohomology

$$H^*(A) \cong k[u, y], \quad for\ A = \langle a, b\rangle \cong \mathbb{Z}/p \times \mathbb{Z}/p,$$

where $u = c_1(e_a)$ is the first Chern class of a non zero linear representation $e_a : A \to \langle a\rangle \to \mathbb{C}^\times$ and $y = c_1(e_b)$ is defined similarly.

At first we consider the case $B = \langle b\rangle \cong \mathbb{Z}/p$ with $H^*(B) \cong k[y]$. The outer automorphism $Out(B) \cong \mathbb{F}_p^*$ and its simple modules are written $S_i = k\{y^i\}$ for $0 \leq i \leq p-2$. (Here we use the notation that $R\{x, y, ...\}$ is the R-free module generated by $x, y, ...$) The summand $L(1, i)$ is defined as $X(S(B, B, S_i)) = X_{S(B,B,S_i)}$ and $H^*(L(1, i)) \cong k[Y]\{y^i\}$ where $Y = y^{p-1}$. Hence we can decompose

$$B\langle b\rangle \cong \vee_{i=0}^{p-2} L(1, i), \quad H^*(L(1, i)) \cong k[Y]\{y^i\} \quad with\ Y = y^{p-1}.$$

The outer automorphism $Out(A) \cong GL_2(k)$ and its simple modules are written as $S(A)^i \otimes det^j$ for $0 \leq i \leq p-1, 0 \leq j \leq p-2$. Here

$$S^i(A) = H^i(A) \cong k\{y^i, y^{i-1}u, ..., u^i\}, \quad dim(S(A)^i) = i + 1$$

and det is the determinate representation. Let us write $L(1, i) = X_{S(P,\mathbb{Z}/p,S_i)}$, that is the image of the same named component by the

split projections $A \to \langle ab^{\lambda}\rangle$, $\langle b\rangle$ $0 \le \lambda \le p-1$. Note that $Tr_A^P(x) = 0$ for all x. Hence we can show (e.g., Harris-Kuhn [Ha-Ku])

$$BA \cong \vee_{i,q}(i+1)\tilde{X}_{i,q} \vee_{i\neq 0} (i+1)L(1,i)$$

where $\tilde{X}_{i,q} = X_{S(A,A,S(A)^i\otimes det^q)}$ for $0 \le i \le p-1$, $0 \le q \le p-2$, and $L(1,p-1) = L(1,0)$. However its decomposition of cohomology $H^*(A)$ is complicated.

For $j = (p-1)+i$ with $0 \le i \le p-2$, we consider

$$S(A)^j = k\{y^j,\ y^{j-1}u, ..., y^p u^{i-1}, T(A)^i, y^{i-1}u^p, ..., yu^{j-1}, u^j\}$$
$$with \quad T(A)^i = k\{y^{p-1}u^i,\ y^{p-2}u^{i+1},\ ...,\ y^i u^{p-1}\}.$$

(Note $S(A)^{p-1} = T(A)^0$.) Let $d_2 = y^p u - yu^p \in H^*(A)^{SL_2(k)}$ so that

$$y^{j-1}u = y^{i-1}u^p,\ ...,\ y^p u^{i-1} = yu^{j-1} \quad mod(d_2).$$

Then $S(A)^j \cong \tilde{S}(A)^i \oplus T(A)^i\ mod(d_2)$, where

$$\tilde{S}(A)^j = k\{y^j\} \oplus k\{y^{j-1}u, ..., , y^p u^{i-1}\} \oplus k\{u^j\}.$$

We can see $\tilde{S}(A)^j\ mod(d_2)$ is a $GL_2(k)$-module, and hence $T(A)^i \cong S(A)^j/(\tilde{S}(A)^j, (d_2))$ is also $GL_2(k)$-module (see [Hi-Ya1]). Moreover, we have

Lemma 4.1. *(Lemma 4.3 in [Hi-Ya1]) There is an $Out(E)$-module isomorphism $T(A)^i \cong S(A)^{p-1-i} \otimes det^i[2i]$ where $[2i]$ means the ascending degree $2i$ operation.*

Hereafter we use notation such that $A \ominus B = C$ means $A = B \oplus C$.

Theorem 4.2. *We have an $Out(A)$-module decomposition*

$$H^*(A) \leftrightarrow k[d_2] \otimes ((k[\bar{C}] \otimes (\oplus_{i=0}^{p-2} S(A)^i \oplus T(A)^i)) \ominus k\{\bar{C}\})$$

where $Out(A)$ acts trivially on d_2 and $\bar{C}$, and $|\bar{C}| = 2(p-1)$.

Remark. The above theorem is proved in [Hi-Ya1], by using the map $q : E \to E/\langle c\rangle \to A$ where $E = p_+^{1+2}$ and $\langle c\rangle$ is its center (see §6). Then we can take

$$grH^*(A) \cong Im(q^*) \oplus H^*(A)\{d_2\} \cong Im(q^*) \otimes k[d_2].$$

The right hand side above is the module in the theorem. There is an element $C \in H^{2(p-1)}(E)$ such that $C \notin Im(q^*)$ but $Cx \in Im(q^*)$ for $x \in H^+(A)$. We define $\bar{C}x = (q^*)^{-1}(Cx)$. (Hence $\bar{C}$ itself does not exists in $grH^*(A)$.) For example, $\bar{C}y = Yy, \bar{C}u = Uu$ with $U = u^{p-1}$, and $\bar{C}^2 = Y^2 + U^2 - YU$, $\bar{C}^3 = Y^3 + U^3 - Y^2U\ mod(d_2)$.

Let us write $\tilde{H}_{i,q} \cong (i+1)H^*(\tilde{Y}_{i,q})$ is a summand of $H^*(A)$ which is the sum of all sub and quotient modules isomorphic to $S(A)^i \otimes det^q$. Let us write $D_2 = d_2^{p-1}$. Note that we use $k[\bar{C}] \ominus k\{\bar{C}\} \cong k[\bar{C}^2]\{1, \bar{C}^3\}$.

Corollary 4.3. *For $0 \le q \le p-2$, we have*

$$\tilde{H}_{i,q} \leftrightarrow \begin{cases} k[\bar{C}^2, D_2]\{1, \bar{C}^3\} & if\ i = q = 0, \\ k[\bar{C}^2, D_2]\{1, \bar{C}^3\}\{d_2^q\} & if\ i = 0,\ q > 0, \\ k[\bar{C}, D_2] \otimes (S(A)^i \otimes d_2^q \oplus T(A)^{p-1-i} \otimes d_2^{i+q}) & otherwise. \end{cases}$$

Since $H^*(L(1,i)) \subset H^*(\tilde{Y}_{i,0})$, (in fact $H^*(L(1,i)$ does not contain d_2^q), we have

$$\tilde{Y}_{i,q} \cong \begin{cases} \tilde{X}_{i,q} & if\ q \ne 0\ or\ (i,q) = (0,0) \\ \tilde{X}_{i,0} \vee L(1,i) & if\ q = 0,\ i \ne 0. \end{cases}$$

Let us write $\tilde{\mathbb{C}\mathbb{B}} = k[\bar{C}, D_2] \cong k[Y, D_2]$. Then we get

Theorem 4.4. *([Hi-Ya1]) We have $H^*(\tilde{X}_{0,q}) \cong \tilde{H}_{0,q}$. For $i \ge 1$, we have*

$$(i+1)H^*(\tilde{X}_{i,q}) \cong \begin{cases} \tilde{\mathbb{C}\mathbb{B}} \otimes (S(A)^i\{D_2\} \oplus T(A)^{p-1-i} \otimes d_2^i), & q = 0 \\ \tilde{\mathbb{C}\mathbb{B}} \otimes (S(A)^i\{d_2^q\} \oplus T(A)^{p-1-i} \otimes d_2^{i+q}) & q \ne 0. \end{cases}$$

In particular, the space $\tilde{X}_{p-1,q}$ (which is written as $L(2,q)$ e.g., in [Mi-Pr]),

$$pH^*(L(2,q)) \cong \begin{cases} \tilde{\mathbb{C}\mathbb{B}} \otimes (S(A)^{p-1}\{D_2\}), & q = 0 \\ \tilde{\mathbb{C}\mathbb{B}} \otimes (S(A)^{p-1}\{d_2^q\}) & q \ne 0. \end{cases}$$

Proof. We only need to prove the case $i \ne 0$ and $q = 0$. First note

$$(i+1)H^*(\tilde{X}_{i,0}) \cong \tilde{H}_{i,q} \ominus (i+1)H^*(L(1,2).$$

Using Corollary 4.3, we can compute

$$(i+1)H^*(\tilde{X}_{i,0}) \ominus (\tilde{\mathbb{C}\mathbb{B}} \otimes T(A)^{p-1-i} \otimes d_2^i)$$

$$\cong \tilde{\mathbb{C}\mathbb{B}} \otimes (S(A)^i) \ominus (i+1)k[Y]\{y^i\} \cong (\tilde{\mathbb{C}\mathbb{B}} \ominus k[Y]) \otimes S(A)^i.$$

Here we identify $(i+1)k\{y^i\} \cong k\{y^i, y^{i-1}u, ..., u^i\} \cong S(A)^i$. The result follows from

$$\tilde{\mathbb{C}\mathbb{B}} \ominus k[Y] \cong k[Y, D_2] \ominus k[Y] \cong k[Y, D_2]\{D_2\} \cong \tilde{\mathbb{C}\mathbb{B}}\{D_2\}.$$

□

We recall the Dickson algebra $\mathbb{DA}$, namely,

$$\mathbb{DA} = k[y,u]^{GL_2(k)} = k[D_1, D_2]$$

where $D_1 = Y^p + V$ and $V = D_2/Y$ (see §5, 6 below). Note that $\bar{C}^p = V + Y^p = D_1\ mod(d_2)$. Hence we can identify (as a free $\mathbb{DA}$-module)

$$\tilde{\mathbb{C}\mathbb{B}} = \mathbb{DA}\{1, \bar{C}, ..., \bar{C}^{p-1}\}.$$

Note $H^*(\tilde{X}_{0,0}) \not\cong \mathbb{DA}$. In fact, we have

Lemma 4.5.
$$H^*(\tilde{X}_{0,0}) \cong k[\bar{C}^2, D_2]\{1, \bar{C}^3\} \cong \mathbb{DA}\{1, D_1\bar{C}, \bar{C}^2, ..., \bar{C}^{p-1}\}.$$

Proof. We see the last isomorphism
$$\mathbb{CB} \ominus k\{\bar{C}\} \otimes k[D_2] \cong \mathbb{DA}\{1, \bar{C}, ..., \bar{C}^{p-1}\} \ominus k\{\bar{C}\} \otimes k[D_2]$$
$$\cong \mathbb{DA}\{1, \bar{C}^2, ..., C^{p-1}\} \oplus A \quad with\ A = \mathbb{DA}\{\bar{C}\} \ominus k[D_2]\{\bar{C}\}.$$
Here $A \cong (\mathbb{DA} \ominus k[D_2])\{\bar{C}\} \cong \mathbb{DA}\{D_1\bar{C}\}$. Thus we have the result. □

Examples. For $p = 3$, (see Corollary 5.2 in [Ya2]) we have
$$H^*(\tilde{X}_{0,0}) \cong \tilde{H}_{0,0} \cong H^*((A : SD_{16})) \cong H^*(A)^{SD_{16}}$$
$$\cong \mathbb{DA}\{1, \tilde{C}D_1, \tilde{C}^2\} \cong \mathbb{DA}\{1, \tilde{C}^2, \tilde{C}^4\}.$$

5. METACYCLIC GROUPS FOR $p \geq 3$

For $p \geq 5$, groups P with $rank_p P = 2$ are classified by Blackburn (see Thomas [Th], Dietz-Priddy [Di-Pr]). They are metacyclic groups, groups $C(r)$ and $G(r', e)$ (see sections 5,7 below for the definitions). In this section, we consider metacyclic p groups P for $p \geq 3$
$$0 \to \mathbb{Z}/p^m \to P \to \mathbb{Z}/p^n \to 0.$$
These groups are represented as
$$P = \langle a, b | a^{p^m} = 1, a^{p^{m'}} = b^{p^n}, [a, b] = a^{rp^\ell}\rangle \quad r \neq 0\ mod(p) \quad (5.1).$$
It is known by Thomas [Th], Huebuschmann [Hu] that $H^{even}(P; \mathbb{Z})$ is multiplicatively generated by Chern classes of complex representations. Let us write
$$\begin{cases} y = c_1(\rho), & \rho : P \to P/\langle a\rangle \to \mathbb{C}^* \\ v = c_{p^{m-\ell}}(\eta), & \eta = Ind_H^P(\xi), \quad \xi : H = \langle a, b^{p^{m-\ell}}\rangle \to \langle a\rangle \to \mathbb{C}^* \end{cases}$$
where ρ, ξ are nonzero linear representations. Then $H^{even}(P; \mathbb{Z})$ is generated by
$$y,\ c_1(\eta), c_2(\eta), ..., c_{p^{m-\ell}}(\eta) = v.$$
(Lemma 3.5 and the explanation just before this lemma in [Ya1].) We can see (the last equation in the proof of Theorem 5.45 in [Ya1])
$$c_1(\eta) = 0, ..., c_{p^{m-\ell}-1}(\eta) = 0 \quad in\ H^*(P) = H^*(P; \mathbb{Z})/(p, \sqrt{0}).$$
By using Quillen's theorem and the fact that P has just one conjugacy class of maximal abelian p-subgroups, we can prove

Theorem 5.1. *(Theorem 5.45 in [Ya1]) For any metacyclic p-group P in (5.1) with $p \geq 3$, we have a ring isomorphism*

$$H^*(P) \cong k[y,v], \quad |v| = 2p^{m-\ell} \quad (5.2).$$

We now consider the stable splitting.

(I) Non split cases. For a non split metacyclic groups, it is proved that BP itself is irreducible [Di].

(II) Split cases with $(\ell, m, n) \neq (1,2,1)$. We consider a split metacyclic group. it is written as

$$P = M(\ell, m, n) = \langle a, b | a^{p^m} = b^{p^n} = 1, [a,b] = a^{p^\ell} \rangle$$

for $m > \ell \geq max(m-n, 1)$.

The outer automorphism is the semidirect product

$$Out(P) \cong (p-group) : \mathbb{Z}/(p-1).$$

The p-group acts trivially on $H^*(P)$, and $j \in \mathbb{Z}/(p-1)$ acts on $a \mapsto a^j$ and so acts on $H^*(P)$ as $j^* : v \mapsto jv$. There are $p-1$ simple $\mathbb{Z}/(p-1)$-modules $S_i \cong k\{v^i\}$. We consider the decomposition by idempotens for $Out(P)$. Let us write $Y_i = e_{S_i} BP$ and

$$H^*(Y(S_i)) \cong (dim(S_i)) H^*(Y_i) \subset H^*(P)$$

(in the notation Y_i from Lemma 3.1). Hence we have the decomposition for $Out(P)$-idempotents

$$H^*(Y_i) \cong k[y, V]\{v^i\}, \quad V = v^{p-1}.$$

We assume $P \neq M(1,2,1)$. By Dietz, we have splitting

$$(*) \quad BP \cong \vee_{i=0}^{p=2} X_i \vee \vee_{i=0}^{p-2} \bar{L}(1,i).$$

Here we write $X_i = e_{S(P,P,S_i)} BP$ identifying S_i as the $A(P,P)$ simple module (but not the simple $Out(P)$-module).

The summand $\bar{L}(1,i)$ is defined as follows. (When $n = 1$, $\bar{L}(1,i) = L(1,i)$ defined in §4.) Recall that $H^*(\langle b \rangle) \cong k[y]$. The outer automorphism group is $Out(\langle b \rangle) \cong (\mathbb{Z}/p^n)^*$ and its simple k modules are $S_i' = k\{y^i\}$ for $0 \leq i \leq p-2$. Hence we can decompose

$$B\langle b \rangle \cong \vee_{i=0}^{p-2} \bar{L}(1,i), \quad H^*(\bar{L}(1,i)) \cong k[Y]\{y^i\} \quad with\ Y = y^{p-1}.$$

Next we consider $\bar{L}(1.i)$ as a split summand in BP as follows. (Consider the $A(P,P)$-simple module $S(P, \langle b \rangle, S_i').$) Let $\Phi \in A(P,P)$ be the element defined by the map $\Phi : P \geq P \to \langle b \rangle \subset P$ which induced the isomorphism

$$H^*(P)\Phi \cong H^*(\vee_{i=0}^{p-2} \bar{L}(1,i)) \cong k[y] \subset H^*(Y_0).$$

Thus we can show (since $k[y]$ is invariant under elements in $Out(P)$)

$$(**) \quad Y_i \cong \begin{cases} X_i & i \neq 0 \\ X_0 \vee \vee_{j=0}^{p-2} \bar{L}(1,j) & i = 0. \end{cases}$$

Remark. For groups P, P' with the same $m - \ell$, we have the isomorphism $H^*(P) \cong H^*(P')$ and the Burnside algebras act on the cohomology by the same way. For the splittings $X(P)$ and $X(P')$ (for BP and BP' respectively), we have $H^*(X_i(P)) \cong H^*(X_i(P'))$. But when $P \not\cong P'$, it is known from Nishida [Ni] that $X_i(P) \not\cong X_i(P')$, i.e. they are not stably homotopy equivalent. Similarly $\bar{L}(1,i)$ are different stable homotopy types when n are different.

Theorem 5.2. *Let P be a split metacyclic group with $(\ell, m, n) \neq (1,2,1)$. Then we have*

$$H^*(X_i) \cong \begin{cases} k[y,V]\{v^i\} & i \neq 0 \\ k[y,V]\{V\} & i = 0. \end{cases}$$

Proof. For $i \neq 0$, we have $H^*(Y_i) \cong H^*(X_i)$. For $i = 0$, we see

$$H^*(X_0) \cong H^*(Y_0) \ominus H^*(\vee_{j=0}^{p-2} L(1,j))$$
$$\cong k[y,V] \ominus k[y] \cong k[y,V]\{V\}.$$

□

(III) Split metacyclic group with $(\ell, m, n) = (1,2,1)$.

This case $P = p_-^{1+2}$ and its cohomology is the same as (II). But the splitting is given ([Di], [Di-Pr])

$$BP \cong \vee_{i=0}^{p=2} X_i \vee \vee_{i=0}^{p-2} L(2,i) \vee \vee_{i=0}^{p-2} L(1,i).$$

Let $H = \langle b, a^p \rangle$ the maximal elementary abelian subgroup. The outer automorphism $Out(H) \cong GL_2(k)$ and simple $GL_2(k)$-modules are written as $S(H)^i \otimes det^j$. The summand $L(2,i)$ is defined as

$$L(2,i) = X_{S(P,H,S(H)^{p-1} \otimes det^i)}.$$

Here note $v|_H = u^p - y^{p-1}u$ so that $yv|_H = d_2$. This fact is proved by using the fact that $v|_H$ invariant under the action $a^* : u \mapsto u+y, y \mapsto y$.

Of course there is no map $P \to H$. The space $L(2,i)$ is the transfer $(Tr : BH \to BG)$ image of the same named summand of BH.

By using the double coset formula

$$Tr_H^P(u^{p-1})|_H = \sum_{i=0}^{p-1} (u+iy)^{p-1} = -y^{p-1}$$

taking the generator u in $H^*(\langle b, a^p\rangle) \cong k[y,u]$. The group P has just one conjugacy class H of the maximal abelian p-groups. Hence by Quillen's theorem, we have

$$Tr_H^P(\bar{C}^i d_2^j u^{p-1}) = -Y^i(yv)^j Y \quad in\ H^*(P) = H^*(P;\mathbb{Z})/(p, \sqrt{0}).$$

Next, we consider an element $\Phi \in A(P,P)$ defined by

$$\Phi : P \geq H \overset{a^p \leftrightarrow b}{\cong} H \subset P.$$

Then $\Phi(C^i d_2^j) = -C^i d_2^j$. (So $\Phi^2(C^i d_2^j) = C^i d_2^j$).) Here recall Theorem 4.4

$$H^*(L(2,i)) \cong \begin{cases} \tilde{\mathbb{C}}\mathbb{B}\{\bar{C}d_2^i\} & i \neq 0 \\ \bar{\mathbb{C}}\mathbb{B}\{\bar{C}D_2\} & i = 0. \end{cases}$$

Since $C^i d_2^j = C^i y^j v^j \in H^*(Y_j)$, we see $\Phi(H^*(L(2,j)) \subset H^*(Y_j)$. Thus we have the isomorphism

$$Y_i \cong \begin{cases} X_i \vee L(2,i) & i \neq 0 \\ X_0 \vee L(2,0) \vee \vee_{j=0}^{p-2} L(1,j) & i = 0. \end{cases}$$

To compute cohomology of irreducible components X_i and $L(2,j)$, we recall the Dickson algebra

$$\mathbb{DA} = k[y,u]^{GL_2(\mathbb{Z}/p)} \cong k[D_1, D_2] \quad with\ D_1 = Y^p + V,\ D_2 = YV.$$

We also write (see §6 bellow) the free $\mathbb{DA}$-modules

$$\mathbb{CA} = k[Y,V] \cong \mathbb{DA}\{1, Y, ..., Y^p\},$$
$$\mathbb{CB} = k[Y, D_2] \cong \mathbb{DA}\{1, Y, ..., Y^{p-1}\}.$$

Hence $\mathbb{CA} \cong \mathbb{DA} \oplus \mathbb{CB}\{Y\}$.

Theorem 5.3. *Let $P = M(1,2,1) \cong p_-^{1+2}$. Then we have*

$$H^*(X_i) \cong \begin{cases} \mathbb{CA}\{1, ..., \hat{y}^i, ..., y^{p-2}\}\{v^i\} \oplus \mathbb{DA}\{d_2^i\} & i > 0 \\ \mathbb{CA}\{y, , ..., y^{p-2}\}\{V\} \oplus \mathbb{DA} & i = 0. \end{cases}$$

Proof. Let $i \neq 0$. We see

$$H^*(Y_i) \cong k[y,V]\{v^i\} \cong \mathbb{CA}\{1, y, ..., y^{p-2}\}\{v^i\}.$$

The cohomology of the summand X_i is

$$\begin{aligned} H^*(X_i) &\cong H^*(Y_i) \ominus H^*(L(2,i)) \\ &\cong \mathbb{CA}\{v^i\}\{1, ..., y^{p-2}\} \ominus \mathbb{CB}\{Yd_2^i\} \\ &\cong \mathbb{CA}\{1, ..., \hat{y}^i, ..., y^{p-2}\}\{v^i\} \oplus (\mathbb{CA}\{v^i y^i\} \ominus \mathbb{CB}\{Yd_2^i\}). \end{aligned}$$

Here $v^i y^i = d_2^i$ and $\mathbb{CA} \cong \mathbb{DA} \oplus \mathbb{CB}\{Y\}$, and we have the isomorphism in the theorem for $i \neq 0$.

Next we consider in the case $i = 0$. From Theorem 5.2, we see

$$H^*(Y_0) \ominus H^*(\vee_j L(1,j) \cong k[y,v]\{V\} \cong \mathbb{CA}\{1,...,y^{p-2}\}\{V\}.$$

Hence we have

$$H^*(X_0) \cong H^*(Y_0) \ominus H^*(\vee_j L(1,j)) \ominus H^*(L(2,0))$$
$$\cong \mathbb{CA}\{1,y,...,y^{p-2}\}\{V\} \ominus \mathbb{CB}\{YD_2\} \cong \mathbb{CA}\{y,...,y^{p-2}\}\{V\} \oplus B$$

where

$$B = \mathbb{CA}\{V\} \ominus \mathbb{CB}\{YD_2\} \cong \mathbb{CA} \ominus H^*(L(1,0)) \ominus H^*(L(2,0)).$$

We can see $B \cong \mathbb{DA}$ by the following lemma. □

Lemma 5.4. *Let $M(2) = L(2,0) \vee L(1,0)$ (as the usual notation of the homotopy theory). Then we have*

$$H^*(M(2)) \cong \mathbb{CB}\{Y\}, \quad \mathbb{CA} \cong \mathbb{DA} \oplus H^*(M(2)).$$

Proof. We can compute

$$H^*(M(2)) \cong k[Y] \oplus \mathbb{CB}\{YD_2\} \cong k[Y] \oplus k[Y,D_2]\{YD_2\}$$
$$\cong (k[Y] \oplus k[Y,D_2]\{D_2\})\{Y\} \cong \mathbb{CB}\{Y\} \quad (assumed * > 0).$$

Since $\mathbb{CA} \cong \mathbb{DA} \oplus \mathbb{CB}\{Y\}$, we have the last isomorphism in this lemma. □

6. $C(r)$ groups for $p \geq 3$

The group $C(r)$, $r \geq 3$ is the p-group of order p^r such that

$$C(r) = \langle a,b,a | a^p = b^p = c^{p^{r-2}} = 1, \ [a,b] = c^{p^{r-3}} \rangle$$

for $r \geq 3$ so that $C(3) = p_+^{1+2}$. Hence we have a central extension

$$0 \to \mathbb{Z}/p^{r-2} \to C(r) \to \mathbb{Z}/p \times \mathbb{Z}/p \to 0.$$

For each $r \geq 3$, the cohomology $H^*(C(r))$ is isomorphic to $H^*(C(3))$. Denote $C(3) = p_+^{1+2}$ simply by E. The cohomology of E is well known. In particular recall that ([Lw], [Le1,2],[Te-Ya])

$$(1) \quad H^*(E) \cong (k[y_1,y_2]/(y_1^p y_2 - y_1 y_2^p) \oplus k\{C\}) \otimes k[v].$$

Here y_1 (resp. y_2) is the first Chern class $c_1(e_1)$ (resp. $c_1(e_2)$) for the nonzero linear representation $e_1 : E \to \langle a \rangle \to \mathbb{C}^*$ (resp. $e_2 : E \to \langle b \rangle \to \mathbb{C}^*$). The elements C and v are also represented by Chern classes

$$c_i(Ind_A^E(e)) = \begin{cases} v & for\ i = p \\ C & for\ i = p-1 \end{cases}$$

where $e : A \to \langle c \rangle \to \mathbb{C}^*$ is a non zero linear representation, for any maximal elementary abelian subgroup A. Hence $|y_i| = 2, |C| = 2(p-$

$1), |v| = 2p$. It is well known $Cy_i = y_i^p$, $C^2 = y_1^{2p-2} + y_2^{2p-2} - y_1^{p-1}y_2^{p-1}$. In this paper we write y_i^{p-1} by Y_i, and v^{p-1} by V.

From the Poincare series and formula (1), we get the another expression of $H^*(E)$ (Proposition 9 in [Gr-Le], or [Ya2])

$$(2) \quad H^*(E) \cong k[C,v]\{y_1^iy_2^j | 0 \le i,j \le p-1, (i,j) \neq (p-1,p-1)\}.$$

Let us write $(\mathbb{Z}/p)^2$ by A simply. The E conjugacy classes of A-subgroups are written by

$$A_i = \langle c, ab^i \rangle \ for\ 0 \le i \le p-1, \quad A_\infty = \langle c, b \rangle.$$

For $A = A_i$ some i, if we take y, u as $H^*(A) \cong k[y,u]$, then $C|_A = Y$ and $v|_A = u^p - y^{p-1}u$. The transfer map is given by $Tr_{A_0}^E(y) = 0$ and

$$(3) \quad Tr_{A_j}^E(u^i) = \begin{cases} (jy_1 + y_2)^{p-1} - C & if\ i = p-1 \\ 0 \quad for\ i < p-1 \end{cases}$$

(for $j = \infty$, we have $tr_{A_\infty}^E(u^{p-1}) = y_1^{p-1} - C$).

We first consider the $Out(E)$-module decomposition of $H^*(E)$. Recall that $Out(E) \cong Out(A) \cong GL_2(\mathbb{F}_p)$. The simple modules of $G = GL_2(\mathbb{F}_p)$ are well known. Let us think of A as the natural two-dimensional representation, and det the determinant representation of G. Then there are $p(p-1)$ simple $k[G]$-modules given by $S(A)^i \otimes (det)^q$ for $0 \le i \le p-1, 0 \le q \le p-2$ where

$$S(A)^i = k\{y_1^i,\ y_1^{i-1}y_2,\ ,...,\ y_2^i\} \cong H^{2i}(E).$$

(This $S(A)^i$ is isomorphic to $S(A)^i$ in §4, but take generators y_1 (resp. y_2) for y (resp. u).

Recall that $k\{v\} \cong det$ as $Out(E)$-modules. Note that

$$\mathbb{CA} = k[C,V] \cong H^*(E)^{Out(E)}.$$

For $j = (p-1) + i$ with $0 \le i \le p-2$. Write it

$$H^i(E) \supset T(A)^i, \quad T(A)^i = k\{y_1^{p-1}y_2^i,\ y_1^{p-2}y_2^{i+1},\ ...,\ y_1^iy_2^{p-1}\}.$$

(Note $S(A)^{p-1} = T(A)^0$.) Using the the relation $d_2 = y_1^py_2 - y_1y_2^p = 0$ in $H^*(E)$, we can consider $T(A)^i$ is an $Out(E)$-module such that $T(A)^i \cong S(A)^{p-1-i} \otimes det^i[2i]$ from Lemma 4.1. In fact, from (2), we also have

$$H^*(E) \cong k[C,v] \otimes (\oplus_{i=0}^{p-2}(S(A)^i \oplus T(A)^i)).$$

Hence we have

Theorem 6.1. *(Theorem 4.4 in [Hi-Ya1]) There is a decomposition of $Out(E)$-module such that*

$$H^*(E) \leftrightarrow \mathbb{CA} \otimes (\oplus_{q=0}^{p-2} \oplus_{i=0}^{p-2} (S(A)^i \otimes v^q \oplus T(A)^i \otimes v^q))$$

where $S(A)^i \otimes v^q \cong S(A)^i \otimes det^q, \quad T(A)^i \otimes v^q \cong S(A)^{p-1-i} \otimes det^{i+q}[2i].$

Let us write by $H_{i,q}$ the summand of $H^*(E)$ which is a sum of the all (sub and quotient) modules isomorphic to $S(A)^i \otimes det^q$. (In the notation in §3, $H^*_{i,j} \cong (i+1)H^*(Y_{i,j})$.)

Corollary 6.2. *We have the* $Out(E)$*-module decomposition*

$$H_{i,q} \leftrightarrow \begin{cases} \mathbb{CA} \otimes v^q & for\ i = 0, \\ \mathbb{CA} \otimes T(A)^0 \otimes v^q \quad (T(A)^0 = S(A)^{p-1}) & for\ i = p-1, \\ \mathbb{CA} \otimes (S(A)^i \otimes v^q \oplus T(A)^{p-1-i} \otimes v^{i+q}) & otherwise. \end{cases}$$

(I) $P = C(r)$, $r > 3$.

By Dietz and Priddy, the stable splitting is known. The splitting is given as

$$BP \cong \vee(i+1)X_{i,q} \vee (q+1)L(1,q) \vee pL(1,p-1)$$

where $0 \leq i \leq p-1$, $0 \leq q \leq p-2$ and $L(1,p-1) = L(1,0)$. Transfers from proper subgroups are always zero when $r > 3$. We have

$$Y_{i,q} = \begin{cases} X_{i,q} & q \neq 0 \\ X_{i,0} \vee L(1,i) & q = 0. \end{cases}$$

Theorem 6.3. *Let* $P = C(r)$ *and* $r \geq 4$. *Then*

$$(i+1)H^*(X_{i,q}) \cong \begin{cases} H_{i,q} & if\ q \neq 0 \\ \mathbb{CA} \otimes (S^i(A)\}\{V\} \oplus T^{p-1-i}(A)v^i) & q = 0, i \neq p-1 \\ \mathbb{CA} \otimes S^{p-1}(A)\{V\} & q = 0,\ i = p-1. \end{cases}$$

Proof. We only need to prove $q = 0$. Note that $\mathbb{CA} \cong k[C,V] \cong k[C] \oplus \mathbb{CA}\{V\}$. So we have $\mathbb{CA} \ominus k[C] \cong \mathbb{CA}\{V\}$. Then we can compute as

$$\mathbb{CA} \otimes S^i(A) \ominus (i+1)k[C]\{y^i\}$$
$$\cong \mathbb{CA}\{V\}\{y_1^i, y_1^{i-1}y_2, ..., y_2^i\} \cong \mathbb{CA} \otimes S(A)^i\{V\}.$$

Using this and

$$(i+1)H^*(X_{i,0}) \cong (i+1)H^*(Y_{i,0}) \ominus (i+1)H^*(L(1,i))$$

we can get the theorem (for $q = 0$). □

(II) $C(3) = p_+^{1+2}$.

In this case, the decomposition of cohomology is given in [Hi-Ya1] but it is quite complicated. By Dietz-Priddy, the splitting is given as

$$BP \cong \vee(i+1)_{i,q}X_{i,q} \vee \vee_k(p+1)L(2,q) \vee_q (q+1)L(1,q) \vee pL(1,p-1)$$

where $0 \leq i \leq p-1$ and $0 \leq q \leq p-2$. The different places from $r \geq 4$ are the existence of $L(2,q)$ which are induced from the transfer (see §9 in [Hi-Ya1] for details).

Lemma 6.4. *We have the isomorphisms*

$$(p+1)\oplus_{q=1}^{p-1} H^*(L(2,q)) \cong \mathbb{CB} \otimes (\oplus_{q=1}^{p-1} \oplus_{j\in\mathbb{F}_p\cup\infty} (1)_{q,j})$$
$$\cong \mathbb{CB} \otimes (\oplus_{q=1}^{p-1}(2)_q) \ \textit{where}$$
$$(1)_{q,j} = k\{Tr_{A_j}^E(u^{p-1})d_2(A_j)^q\} \cong k\{((jy_1+y_2)^{p-1} - C)d_2(A_j)^q\},$$
$$(2)_q = \begin{cases} (S(A)^q\{C\otimes v^q\} \oplus T(A)^q \otimes v^q) & 1 \leq q \leq p-2 \\ (S(A)^{p-1} \oplus k\{C\})\{D_2\} & q = p-1 \end{cases}$$

where $d_2(A_j) = v(y_1 + jy_2)$.

Outline of Proof. (See §9 in [Hi-Ya1] for details.) Let $x \in (1)_{q,j} \subset (k\{C\} \oplus S(A)^{p-1}) \otimes d_2^q$. Then for $\mu, \lambda_i \in \mathbb{Z}/p$, the element x is written as

$$\mu C \otimes d_2^q + \sum_i \lambda_i y_1^i y_2^{p-1-i} \otimes d_2^q$$
$$= \mu C y_1^q \otimes v^q + \sum_i \lambda_i y_1^{i+q} y_2^{p-1-i} \otimes v^q$$
$$\in (S(A)^q\{C\otimes v^q\} \oplus T(A)^q \otimes v^q) \subset H_{q,q} \oplus H_{p-1-q,2q}.$$

The last inclusion follows from $T^q \otimes v^q \cong S(A)^{p-1-q} \otimes det^{q+q}$ as $Out(P)$-modules. Hence we see $\oplus_{j\in\mathbb{F}_p\cup\infty}(1)_{q,j} \subset (2)_q$. Since

$$(1)_{q,j}|A_k = \begin{cases} (y_1+jy_2)^{p-1}d_2(A_j)^q & j = k \\ 0 \quad \textit{otherwise}, \end{cases}$$

we have $dim(\oplus_{j\in\mathbb{F}_p\cup\infty}(1)_q) \geq p+1$. Of course (from Theorem 6.1), we have $dim(2)_q = p+1$. Therefore we show that $\oplus_{j\in\mathbb{F}_p\cup\infty}(1)_q = (2)_q$. □

Note that $(p+1)H^*(L(2,q)) \subset H_{q,q} \oplus H_{p-1-q,i+q}$, however each $H^*(L(2,q))$ is contained in either $H^*(Y_{q,q})$ or $H^*(Y_{p-1-i,i+q})$.

Theorem 6.5. *The $Out(E)$-module decomposition*

$$H^*(E) \leftrightarrow \oplus_{i,q} H_{i,q} \cong \oplus_{i,q} \mathbb{CA} \otimes (S(A)^i \otimes v^q \oplus T(A)^{p-1-i} \otimes v^{2q})$$

gives simple $A(E,E)$-modules decomposition by each of the following sums of $Out(E)$-simple modules

$$(1) \quad S(A)^{p-1} \oplus k\{C\}$$
$$(2) \quad S(A)^q\{C\} \otimes v^q \oplus T(A)^q \otimes v^q$$

as one $A(E,E)$-simple module for $1 \leq q \leq p-2$. That is

$(1) \cong S(E, \mathbb{Z}/p, det^0)[2p-2]$, $\quad (2) \cong S(E, A, S(A)^{p-1} \otimes det^q)[2(p-1)q]$.

Outline of Proof. We prove that (1) is a simple $A(E,E)$-module. We consider $\Phi_1, \Phi_2 \in A(E,E)$

$$\Phi_1 : P > A_0 \to \langle c\rangle \overset{w}{\cong} \langle a\rangle \subset A_0 \subset P,$$

$$\Phi_2 : P > A_\infty \to \langle c\rangle \overset{w}{\cong} \langle b\rangle \subset A_\infty \subset P.$$

Then we see $(\Phi_1 - \Phi_2)(C) = y_1^{p-1} - y_2^{p-1} \in S^{p-1}(A)$, and

$$\Phi_1(y_1^{p-1}) = y_2^{p-1} - C \in S^{p-1}(A) \oplus k\{C\}.$$

Let $B = (S^{p-1}(A) \oplus k\{C\})$. Then from the first equation, $B/k\{C\}$ is not an $A(P,P)$-module. From the second one, we see that $B/S^{p-1}(A)$ is also not an $A(P,P)$-module. Hence B is a simple $A(P,P)$-module.

The fact that (2) is isomorphic to a $A(P,P)$ simple module is proved similarly using Φ_1' and Φ_2' defined by

$$\Phi_1' : P > A_0 \overset{a\leftrightarrow c}{\cong} A_0 \subset P, \quad \Phi_2' : P > A_\infty \overset{b\leftrightarrow c}{\cong} A_\infty \subset E.$$

□

Note that $dim(2) = (q+1)+(p-1-q+1) = p+1$. In fact, this is the number of $L(2,q)$ in BE.

Let $i = 0$ or $p-1$. Then we see

$$Y_{i,q} \cong \begin{cases} X_{i,q} & q \neq 0 \\ X_{i,0} \vee L(2,0) \vee L(1,0) & q = 0. \end{cases}$$

Using this we can prove

Theorem 6.6. *(Corollary 10.7 in [Hi-Ya1]) We have for* $1 \le q \le p-2$,

$$H^*(X_{0,0}) \cong \mathbb{DA}, \quad pH^*(X_{p-1,0}) \cong \mathbb{DA} \otimes S(A)^{p-1}\{V\},$$

$$H^*(X_{0,q}) \cong H_{0,q} \cong \mathbb{CA}\{v^q\}, \quad pH^*(X_{p-1,q}) \cong H_{p-1,q} \cong \mathbb{CA}\{S(A)^{p-1} \otimes v^q\}.$$

Proof of the first isomorphism. Recall $H^*(L(2,0) \vee L(1,0)) \cong \mathbb{CB}\{C\}$ from Lemma 5.4. Hence we see

$$H^*(X_{0,0}) \cong \mathbb{CA} \ominus \mathbb{CB}\{C\} \cong \mathbb{DA}.$$

The other cases are proved similarly. □

Examples. See §6 in [Ya2] for $p=3$ case. For the sporadic simple group J_4 and the twisted Chevalley group ${}^2F_4'$, we have the isomorphisms

$$H^*(J_4) \cong H^*(X_{0,0}) \cong \mathbb{DA}, \quad (Green\ [Gr])$$

$$H^*({}^2F_4') \cong H^*(X_{0,0} \vee X_{2,0}) \cong \mathbb{DA}\{1, YV\},$$

$$H^*(E)^{SD_{16}} \cong H^*(X_{0,0} \vee L(2,0) \vee L(1,0)) \cong \mathbb{CA},$$

Theorem 6.7. *(Corollary 10.8 in [Hi-Ya1]) Let* $1 \le i \le p-2$. *Let us write simply*

$$S = S(A)^i \otimes v^q, \quad T = T(A)^{p-1-i} \otimes v^{i+q}.$$

Then we have the isomorphism

$$(1{+}i)H^*(X_{i,q}) \cong \begin{cases} \mathbb{DA} \otimes (S \oplus T\{V\}) & if\ i = q \neq 0,\ 3q \equiv 0\ (mod(p-1)) \\ \mathbb{DA} \otimes S \oplus \mathbb{CA} \otimes T & if\ i = q,\ 3q \not\equiv 0 \\ \mathbb{CA} \otimes S \oplus \mathbb{DA} \otimes T\{V\} & if\ q \equiv -2i \not\equiv 0,\ 3i \not\equiv 0 \\ \mathbb{CA} \otimes S\{V\} \oplus \mathbb{DA} \otimes T\{V\} & if\ q \equiv 0,\ 2i \equiv 0 \\ \mathbb{CA} \otimes S\{V\} \oplus \mathbb{CA} \otimes T & if\ q \equiv 0,\ 2i \not\equiv 0 \\ H_{i,q} \cong \mathbb{CA} \otimes (S \oplus T) & otherwise. \end{cases}$$

Outline of Proof. From the proof of Lemma 5.5, we see

$$\oplus_j Tr^E_{A_j}(H^*(L(2,q)) \subset H_{q,q} \oplus H_{p-1-q,2q}.$$

We prove the first isomorphism. Suppose $3q \equiv 0,\ q \not\equiv 0\ mod(p-1)$. Then (see the proof of Lemma 6.4)

$$\oplus_j Tr^E_{A_j}(H^*(L(2,2q)) \subset H_{2q,2q} \oplus H_{q,q}$$

since $H_{p-1-2q,4q} = H_{q,q}$. Using this we can prove

$$Y_{q,q} \cong X_{q,q} \vee L(2,q) \vee L(2,2q).$$

Hence we can compute

$$(q+1)H^*(X_{q,q}) \cong (\mathbb{CA} \otimes S \ominus (q+1)\mathbb{CB}\{Cd^q\})$$

$$\oplus(\mathbb{CA} \otimes T \ominus (q+1)\mathbb{CB}\{Cd^{2q}\}).$$

Here $d^q = y^q v^q \in S = S^q(A)\{v^q\}$ and $Cd_2^{2q} = (Cy^{2q})v^{2q} \in T = T(A)^{2q}\{v^{2q}\}$. Hence

$$\mathbb{CA} \otimes S \ominus (q+1)\mathbb{CB}\{Cd^q\} \cong (\mathbb{CA} \ominus \mathbb{CB}\{C\})\{S^q(A)v^q\} \cong \mathbb{DA} \otimes S.$$

$$\mathbb{CA} \otimes T \ominus (q+1)\mathbb{CB}\{Cd^{2q}\} \cong (\mathbb{CA} \ominus \mathbb{CB})\{T^{2q}v^{2q}\} \cong \mathbb{DA}\{C^p\} \otimes T.$$

Note $D_1 = C^p + V$. Thus we can prove the first isomorphism. The other isomorphisms are proved similarly. □

We write down here the splitting in the all cases.

Corollary 6.8. *For $1 \le i \le p-2$, there are stable homotopy equivalences*

$$Y_{i,q} \cong \begin{cases} X_{q,q} \vee L(2,q) \vee L(2,2q) \\ \qquad if\ i = q \neq 0,\ 3q \equiv 0\ (mod(p-1)) \\ X_{q,q} \vee L(2,q) \quad if\ i = q,\ 3q \not\equiv 0 \\ X_{i,-2i} \vee L(2,-i) \quad if\ q \equiv -2i \not\equiv 0,\ 3i \not\equiv 0 \\ X_{i,0} \vee L(1,i) \vee L(2,-i) \quad if\ q \equiv 0,\ 2i \equiv 0 \\ X_{i,0} \vee L(1,i) \quad if\ q \equiv 0,\ 2i \not\equiv 0 \\ X_{i,q} \quad otherwise. \end{cases}$$

Example. When $p = 7$ (see §9 in [Ya2]), we see the cohomology

$$H^*(X_{0,0}) \cong \mathbb{DA},\ H^*(X_{6,0}) \cong \mathbb{DA}\{a^3\},\ H^*(X_{4,4}) \cong \mathbb{DA}\{a^2, a^4\},$$
$$H^*(X_{2,2}) \cong \mathbb{DA}\{a, a^5\}.$$

where $a = s^2 \otimes v^2 \in S(2,2)$, $a^5 = s^{10} \otimes v^{10} = s^{10} \otimes v^4 V \in T(2,2)$,... and where $S(i,q) = S, T(i,q) = T$ in the preceding theorem. Here $a^6 = D_2^2$ (page 416 in [Ya2]). Thus we see the cohomology of the exotic finite 7-groups (see §9 in [Ya2]) found by Ruiz and Viruel [Ru-Vi]

$$H^*(RV_3) \cong H^*(X_{0,0} \vee X_{4,4}) \cong \mathbb{DA}\{1, a^2, a^4\},$$
$$H^*(RV_2) \cong H^*(X_{0,0} \vee X_{4,4} \vee X_{6,0} \vee X_{2,2}) \cong \mathbb{DA}\{1, a, a^2, a^3, a^4, a^5\}$$
$$H^*(RV_1) \cong H^*(X_{0,0} \vee X_{6,0} \vee X_{4,4}) \cong \mathbb{DA}\{1, a^2, a^3, a^4\}.$$

Therefore there does not exist even a 7-local finite group G such that $H^*(G) \cong \mathbb{DA}$.

7. $G(r,e)$ for $p \ge 5$

For $p \ge 5$, groups P with $rank_p P = 2$ are classified by Blackburn (see Thomas [Th], Dietz-Priddy [Di-Pr], [Ya1]). They are metacyclic groups, groups $C(r)$ and $G(r',e)$. Throughout this section, we assume $p \ge 5$.

The group $G = G(r,e), r \ge 4$ (and e is 1 or a quadratic non residue modulo p) is defined as

$$\langle a, b, c | a^p = b^p = c^{p^{r-2}} = [b,c] = 1, [a, b^{-1}] = c^{ep^{r-3}}, [a,c] = b\rangle.$$

The subgroup $\langle a, b, c^p\rangle$ is isomorphic to $C(r-1)$. Hence we have the extension

$$1 \to C(r-1) \to G(r,e) \to \mathbb{Z}/p \to 0.$$

Of course $E = C(3) \subset C(r-1) \subset G(r,e)$. By [Ya1], we have an isomorphism

$$H^*(G(r,e)) \cong H^*(E)^{\langle c\rangle}.$$

Indeed, in Theorem 5.29 in [Ya1], we see $H^{even}(G;\mathbb{Z}) \cong (Y_1 \oplus Y_w \oplus C') \otimes C'_p$ and in (5.5), we see $H^{even}(E;\mathbb{Z})^{\langle c\rangle} \cong (Y_1 \oplus Y_w \oplus C) \otimes C_p$. Here we can show

$$C_p/p \cong C'_p/p, \quad C/p \cong \begin{cases} C'/p & for\ r=4 \\ C'/(p,c_1), \quad (c_1 : nilpotent)\ for\ r \geq 5. \end{cases}$$

The invariant ring $H^*(C(3))^{\langle c\rangle}$ is multiplicatively generated by

$$y_1,\ C,\ v,\ y_2^i w \quad where\ w = y_2^p - y_1^{p-1}y_2, \quad 0 \leq i \leq p-3$$

since $c^* : y_2 \mapsto y_2 + y_1$ and $C^2 = Y_1^2 + y_2^{p-2}w$. Hence we have

Lemma 7.1. *We have an isomorphism*

$$(1) \quad H^*(G(r,e)) \cong (k[y_1] \oplus k[y_2]\{w\} \oplus k\{C\}) \otimes k[v]$$

where the multiplications are given by $y_1 w = 0$, $Cy_1 = y_1^p$, $w^2 = y_2^p w$ and $Cw = y_2^{p-1}w$, Thus we also have the isomorphism

$$(2) \ H^*(G(r,e)) \cong \mathbb{CA}(\oplus_{q=0}^{p-2}(k\{1,y_1,...,y_1^{p-1}\}\{v^q\} \oplus k\{1,y_2,...,y_2^{p-3}\}\{wv^q\}).$$

Here we note that

$$H^*(E)^{\langle c\rangle} \cap \oplus_{i=0}^{p-2} S(A)^i \cong k\{1,y_1,...,y_1^{p-2}\},$$
$$H^*(E)^{\langle c\rangle} \cap \oplus_{i=0}^{p-2} T(A)^i \cong k\{y_1^{p-1}\} \oplus k\{1,y_2,...,y_2^{p-3}\}\{w\}.$$

Let us write $w_{i+1} = y_2^i w$ (so $w_1 = w$) and

$$S(G) = k\{1,y_1,...,y_1^{p-2}\}, \quad T(G) = k\{y_1^{p-1}, w_1, ..., w_{p-2}\}$$

so that $H^*(G(r,e)) \cong \mathbb{CA} \otimes (\oplus_i (S(G) \oplus T(G))\{v^i\})$.

For groups $G' = G(r,e), E, ...$, let us write by $Y_{i,j}(G')$ (and $X_{i,j}(G')$) the decomposition component for BG'. Then from Corollary 6.2, we have

Lemma 7.2. *We have additively*

$$H^*(G(r,e)) \cong \oplus_{i,q} H^*(Y_{i,q}(E)) \qquad with$$

$$H^*(Y_{i,q}(E)) \cong \begin{cases} \mathbb{CA} \otimes v^q & for\ i=0, \\ \mathbb{CA} \otimes k\{y_1^{p-1} \otimes v^q\} & for\ i=p-1, \\ \mathbb{CA} \otimes (k\{y_1^i \otimes v^q, w_{p-1-i} \otimes v^{i+q}\}) & otherwise \end{cases}$$

where $0 \leq i \leq p-1$ and $0 \leq q \leq p-2$.

The outer automorphism is $Out(P) \cong (p-group) : (\mathbb{Z}/2 \times \mathbb{Z}/(p-1))$ (see [Di-Pr] for details). Here the action $i \in \mathbb{Z}/2$ induces $i : a \mapsto a^{-1}$ and $k \in \mathbb{Z}/(p-1)$ induces $k : c \mapsto c^k$. Hence

$$i^* : \begin{cases} y_1 \mapsto -y_1 \\ y_2 \mapsto -y_2, \end{cases} \qquad k^* : \begin{cases} v \mapsto kv \\ y_2 \mapsto ky_2. \end{cases}$$

All simple $\mathbb{Z}/2 \times \mathbb{Z}/(p-1)$-modules are represented as

$$k\{v^i\}, \quad k\{y_1 v^i\} \quad 0 \le i \le p-2.$$

Using this and Lemma 7.2 (2), we get

Lemma 7.3. *Let $P = G(r,e)$ with $r \ge 4$. Then we have $Out(P)$-module decomposition*

$$H_{i,q} \leftrightarrow H^*(Y_{i,q}(P)) \cong \begin{cases} \oplus_{j=even} H^*(Y_{j,q}(E)) & if \ i=0 \\ \oplus_{j=odd} H^*(Y_{j,q}(E)) & if \ i=1 \end{cases}$$

where $0 \le i \le 1$, $0 \le j \le p-1$ and $0 \le q \le p-2$.

(I) The case $P = G(r,e)$ and $r > 4$.
The stable splitting is given by Dietz-Priddy [Di-Pr]

$$BG(r,e) \cong \vee_{i,q} X_{i,q}(G(r,e)) \vee \vee_q X_{p-1,q}(C(r-1)) \vee \vee_q L(1,q)$$

where $i \in \mathbb{Z}/2$ and $0 \le q \le p-2$.
From Theorem 6.3, (for $r-1 \ge 4$) recall

$$pH^*(X_{p-1,q}(C(r-1)) \cong \begin{cases} \mathbb{CA} \otimes S(A)^{p-1}\{V\} & if \ q=0 \\ \mathbb{CA} \otimes S(A)^{p-1} v^q & if \ 0<q<p-1. \end{cases}$$

This summand induced from the following transfer. Recall $[a,c]=b$ in $G(r,e)$ and $c^*(y_2) = y_2 + y_1$ in $H^*(\langle a,b,c^p \rangle) = H^*(C(r-1))$. Hence we can compute

$$Tr^G_{C(r-1)}(y_2^{p-1})|_{C(r-1)} = \sum_j (y_2 - jy_1)^{p-1} = -y_1^{p-1}$$

which implies that $Tr^G_{C(r-1)}(y_2^{p-1}) = -y_1^{p-1}$ since $H^*(P) \subset H^*(C(r-1))$. Define $\Phi \in A(P,P)$ by

$$\Phi : P > C(r-1) \overset{a \leftrightarrow b}{\to} C(r-1) \subset P.$$

Then we have

$$\Phi(v^q y_1^{p-1}) = Tr^P_{C(r-1)}(v^q y_2^{p-1}) = -v^q y_1^{p-1}.$$

Hence $\Phi(H^*(Y_{p-1,q}(E))) \cong \mathbb{CA}\{y_1^{p-1}v^q\} \cong H^*(Y_{p-1,q}(E))$. Thus we have

$$Y_{i,q} \cong \begin{cases} X_{0,0} \vee (X_{p-1,0}(C(r-1)) \vee \vee_{j=ev}^{p-3} L(1,j) & if \ i=q=0 \\ X_{0,q} \vee X_{p-1,q}(C(r-1)) & if \ i=0, \ q \ne 0 \\ X_{1,0} \vee \vee_{j=odd} L(1,j) & if \ i=1, q=0 \\ X_{1,q} & if \ i=1, \ q \ne 0. \end{cases}$$

Theorem 7.4. *For $r > 4$, we have*

$$H^*(X_{i,q}(G(r,e)) \cong \begin{cases} H^*((\vee_{j=ev}^{p-3} X_{j,0}(C(r-1))) \vee L(1,0)) & if\ i=q=0 \\ H^*(\vee_{j=ev}^{p-3} X_{j,q}(C(r-1))) & if\ i=0, q\neq 0 \\ H^*(\vee_{j=odd}^{p-2} X_{j,q}(C(r-1)) & if\ i=1 \end{cases}$$

$$\cong \begin{cases} \mathbb{CA}\otimes(k\{1\}\oplus\oplus_{j=ev>0}^{p-3} k\{y_1^j V, w_{-j}v^j\}) & if\ i=q=0 \\ \mathbb{CA}\otimes(\oplus_{j=ev}^{p-3} k\{y_1^j v^q, w_{-j}jv^{j+q}\}) & if\ i=0, q\neq 0 \\ \mathbb{CA}\otimes(\oplus_{j=odd}^{p-2} k\{y_1^j V, w_{-j}v^j\}) & if\ i=1, q=0 \\ \mathbb{CA}\otimes(\oplus_{j=odd}^{p-2} k\{y_1^j v^q, w_{-j}v^{j+q}\}) & if\ i=1, q\neq 0. \end{cases}$$

where $w_{-j} = w_{p-1-j}$.

Proof. We will prove the case $i = q = 0$, and other cases are proved similarly. From the above isomorphism for $Y_{i,q}$, we have

$$H^*(X_{0,0}) \cong H^*(Y_{0,0}) \ominus H^*(X_{p-1,0}(C(r-1))) \ominus \oplus_{j=ev}^{p-3} H^*(L(1,j)).$$

Using Lemma 7.3, we see $H^*(Y_{0,0}) \cong \oplus_{j:ev}^{p-1} H^*(Y_{j,0}(C(r-1))$. Hence we can write $H^*(X_{0,0}) \cong A \oplus B$ with

$$A = \oplus_{j=ev}^{p-3} H^*(Y_{j,0}(C(r-1)) \ominus \oplus_{j=ev}^{p-3} H^*(L(1,j)),$$
$$B = H^*(Y_{p-1,0}(C(r-1)) \ominus H^*(X_{p-1,0}(C(r-1)).$$

Here we have

$$A \cong \oplus_{j=ev}^{p-3} H^*(X_{j,0})(C(r-1)) \quad and \quad B \cong H^*(L(1,0)).$$

Thus we have the first isomorphism in the theorem for $i = q = 0$.

The second isomorphism follows from

$$H^*(X_{0,0}) \cong A \oplus H^*(L(1,0)) \cong \mathbb{CA}\{1\} \oplus \oplus_{0<j=ev}^{p-3} H^*(X_{j,0}(C(r-1)).$$

Here we used $H^*(X_{0,0}(C(r-1))) \oplus H^*(L(1,0)) \cong \mathbb{CA}\{1\}$. $\square$

(II) $r = 4$.

In this case cohomology is the same as (I). However the stable splitting is not same as (I) and it is also given by Dietz and Priddy Di-Pr].

$$BG(r,e) \cong \vee_{i,q} X_{i,q}(G(r,e)) \vee \vee_q X_{p-1,q}(C(r-1))$$
$$\vee_q L(2,q) \vee \vee_q L(1,q)$$

where $i \in \mathbb{Z}/2$ and $0 \leq q \leq p-2$.

The problems are only to see that these $H^*(L(2,q))$ go to what $H^*(Y_{i,q'})$. Let us consider $\Phi \in A(P,P)$ such that

$$\Phi : P > E > \langle a, c^p\rangle \overset{a\leftrightarrow c^p}{\to} \langle a, c^p\rangle \subset P.$$

Then we can compute (for $d_2 = y_1 v$)

$$\Phi(d_2^q y_1^{p-1}) = Tr_E^P Tr_{\langle a,c^p\rangle}^E (d_2^q u^{p-1}) = Tr_E^P(d_2^q(y_2^{p-1} - C))$$

$$= Tr_E^P(d_2^q(y_2^{p-1} - C))|_E = -d_2^q y_1^{p-1}$$

from (3) in §6 and the arguments before Lemma 7.4. This means $y_1^{q+p-1}v^k$ is in the image from $H^*(L(2,q)) \subset H^*(\langle a, c^p\rangle)$. Note if q is even, then $y_1^{k+p-1}v^q \in H^*(Y_{0,q})$, otherwise it is in $H^*(Y_{1,q})$. Hence we see

$$Tr^P_{\langle a,c^p\rangle} H^*(L(2,q)) \subset \begin{cases} H^*(Y_{0,q}) & q = even \\ H^*(Y_{1,q}) & q = odd. \end{cases}$$

In particular, note that

$$D_2 y_1^{p-1} \in H^*(Y_{p-1,0}(E)) \subset H^*(Y_{0,0}(P))$$

is in the image from $H^*(L(2,0))$. However note $D_2C \in H^*(Y_{0,0}(E)) \subset H^*(Y_{0,0}(P))$ is not in the image from $H^*(L(2,0))$. while it is so in $H^*(Y_{0,0}(E))$.

Thus we have

$$Y_{i,q} \cong \begin{cases} X_{0,0} \vee (X_{p-1,0}(E) \vee L(2,0) \vee L(1,0)) \vee \vee_{j=ev>0}^{p-3} L(1,j) \\ \qquad\qquad if\ i = q = 0 \\ X_{0,q} \vee X_{p-1,q}(E) \vee L(2,q) \quad if\ i = 0,\ q = ev \neq 0 \\ X_{0,q} \vee X_{p-1,q}(E) \quad if\ i = 0,\ q = odd \neq 0 \\ X_{1,0} \vee \vee_{j=odd} L(1,j) \quad if\ i = 1, q = 0 \\ X_{1,q} \quad if\ i = 1,\ q = ev \neq 0. \\ X_{1,q} \vee L(2,q) \quad if\ i = 1,\ q = odd \neq 0. \end{cases}$$

Then we have

Theorem 7.5. *For $P = G(4,e)$, the cohomology $H^*(X_{i,q})$ is isomorphic to*

$$\cong \begin{cases} \mathbb{CA} \otimes (k\{1\} \oplus \oplus_{0<j=ev}^{p-3} k\{y_1^j V, w_{-j}v^j\}) \quad if\ i = q = 0 \\ \mathbb{CA} \otimes (\oplus_{0<j=ev\neq q}^{p-3} k\{y_1^j v^q, w_{-j}v^{j+q}\}) \\ \qquad \oplus \mathbb{DA}\{y_1^q v^q\} \oplus \mathbb{CA}\{w_{-q}v^{2q}\} \quad if\ i = 0, q = even \neq 0 \\ \mathbb{CA} \otimes (\oplus_{j=ev}^{p-3} k\{y_1^j v^q, w_{-j} j v^{j+q}\}) \quad if\ i = 0, q = odd \neq 0 \\ \mathbb{CA} \otimes (\oplus_{j=odd}^{p-2} k\{y_1^j V, w_{-j}v^j\}) \quad if\ i = 1, q = 0 \\ \mathbb{CA} \otimes (\oplus_{j=odd}^{p-2} k\{y_1^j v^q, w_{-j}v^{j+q}\}) \quad if\ i = 1, q = even \neq 0 \\ \mathbb{CA} \otimes (\oplus_{j=odd\neq q}^{p-3} k\{y_1^j v^q, w_{-j}v^{j+q}\}) \\ \qquad \oplus \mathbb{DA}\{y_1^q v^q\} \oplus \mathbb{CA}\{w_{-q}v^{2q}\} \quad if\ i = 1, q = odd \neq 0. \end{cases}$$

Proof. When $i = q = 0$, the isomorphism follows from

$$H^*(X_{0,0}(G(4,e)) \cong H^*(X_{0,0}(G(r,e))) \quad for\ r > 4.$$

This fact is shown from the decomposition $Y_{0,0}$ above and

$$H^*(X_{p-1,0}(C(r-1)) \cong H^*(X_{p-1,0}(C(3)) \vee L(2,0)) \quad for\ r > 4.$$

When $i = 0, q = ev \neq 0$, the fact

$$H^*(X_{0,q}(G(4,e)) \cong H^*(X_{0,q}(G(r,e')) \ominus H^*(L(2,q))$$

implies the isomorphism in the theorem. The other cases are also seen similarly. □

8. Relations among BP with $rank_pP = 2$.

In this section, we see Theorem 1.1 in the introduction. For a group G with $rank_pG = 2$, let us write by $X_{i,q}(G)$ (or $X_i(G)$ for a metacyclic group) the corresponding irreducible stable homotopy summand.

Recall that a non-dominant summand X is the irreducible summand corresponding to an $A(G,G)$-simple module $S(G,Q,V)$ for a proper subgroup Q and a simple $Out(Q)$-module V. From Dietz-Priddy, the following lemma is immediate.

Lemma 8.1. *Let $G = C(r)$ (or $G(r+1,e)$) for $r \geq 3$. Then for $0 \leq q \leq p-2$, a non-dominant summand is $L(1,q)$, $L(2,q)$ (or $X_{p-1,q}(C(r))$ for $G = G(r+1,e)$).*

Let us use the notation such that for stable homotopy spaces A, B, the notation $A \cong_H B$ means $H^*(A) \cong H^*(B)$ as graded modules. Theorem 1.1 in the introduction is a immediate consequence of the above lemma and the following theorem about dominant summands.

Theorem 8.2. *Let $G = C(r)$ (or $G(r+1,e)$) for $r \geq 3$. Given $0 \leq i \leq p-1$ (or $i = 0$ or 1) and $0 \leq q \leq p-2$, there are a_j, b_k, c which are 0 or 1 such that we have the isomorphism*

$$X_{i,q}(G) \cong_H \vee_{j=0}^{p-1} a_j X_{j,q}(E) \vee \vee_{k=0}^{p-2} b_k L(2,k) \vee cL(1,0) \quad (*)$$

In particular, $c = 1$ if and only if $i = q = 0$ and $G = G(r+1,e)$.

Proof. We first consider $G = C(r)$, for $r > 3$. Since $C(r) \cong_H E$, we see $Y_{i,q}(C(r)) \cong_H Y_{i,q}(E)$. Let $q = 0$. Then from the formula for $Y_{i,q}$ just before Lemma 6.3, $Y_{i,0}(C(r)) \cong X_{i,0}(C(r)) \vee L(1,i)$.

On the other hand, from Corollary 6.8, we see

$$Y_{i,0}(E) \cong_H X_{i,0}(E) \vee L(1,i) \vee bL(2,-i) \quad for\ b = 0\ or\ 1.$$

Hence $X_{i,0}(C(r)) \cong_H X_{i,0}(E) \vee bL(2,-i)$, and $(*)$ is satisfied in this case, in particular note $c = 0$. The case $q \neq 0$ is shown similarly by using Corollary 6.8

$$Y_{i,q}(C(r)) \cong_H X_{i,q}(E) \vee b'L(2,i) \vee b''L(2,-i) \quad b', b'' = 0\ or\ 1.$$

The case $G = G(r+1,e), r > 3$ is immediate from the first isomorphism in Theorem 7.4 and the result for $C(r)$. We also note $c = 1$ if and only if $i = q = 0$. Hence we can write

$$X_{i,q}(G(r+1,e)) \cong_H \vee \vee_{j=0}^{p-2} a_j X_{j,q}(E) \vee \vee_{k=0}^{p-2} b_k L(2,k) \vee cL(1,0).$$

where $0 \le a_j, b_k, c \le 1$.

The fact $0 \le b_k \le 1$ is shown by the following arguments. Note if $Y_{j,q}(E)$ contains $b'L(2,k)$, then $b' = 1$ and $k = j$ or $k = -j$ from Corollary 6.8. Therefore $\vee_{j=0}^{p-2} Y_{j,q}(E)$ contains $bL(2,k)$ for $b \le 2$. Suppose $b_k = 2$ and $k \ne 0$. Since $Y_{j,q}(E)$ contains $L(2,j)$ only when $j = q$, we can assume $k = q$. However $Y_{p-1-q,q}(E)$ does not contain $L(2,q)$ from Corollary 6.8. Hence $b_k \ne 2$.

Let $G = G(4,e)$. First note that $Y_{i,q}(G(r+1,e)) \cong_H Y_{i,q}(G(4,e))$. When $i = q = 0$, the isomorphism $(*)$ is immediate from $X_{0,0}(G(4,e)) \cong_H X_{0,0}(G(r+1,e))$. When $i = 0, q = ev \ne 0$, we recall

$$H^*(X_{0,q}(G(4,e)) \cong H^*(X_{0,q}(G(r+1,e)) \ominus H^*(L(2,q))$$

Here $X_{0,q}(G(r+1,e)) \cong_H \vee_{j=even}^{p-3} X_{j,q}(C(r))$. The last space contains

$$X_{q,q}(C(r)) \cong_H X_{q,q}(E) \vee L(2,q) \vee bL(2,2q) \quad for\ b = 0,\ or\ 1$$

from Corollary 6.8, which implies the isomorphism in the theorem. The other cases are also seen similarly. □

Example. Let $p = 7$ and $r > 3$. We see from Corollary 6.8

$$X_{5,2}(C(r)) \cong Y_{5,2}(C(r)) \cong_H Y_{5,2}(E) \cong X_{5,2}(E) \vee L(2,1).$$

We have $H^*(Y_{5,2}(E)) \cong \mathbb{CA}\{y_1^5 v^2, w_1 v\}$ and $H^*(L(2,1)) \cong \mathbb{CB}\{u^{p-1} d_2^1\}$ which maps to $w_1 v$ by the transfer. Hence

$$H^*(X_{5,2}(E)) \cong H^*(Y_{5,2}(E)) \ominus H^*(L(2,1) \cong \mathbb{CA}\{y_1^5 v^2\} \oplus \mathbb{DA}\{w_1 v\}$$

by using $\mathbb{CA} \ominus \mathbb{CB}\{C\} \cong \mathbb{DA}$. For the case $X_{6,0}(E)$, we have $H^*(Y_{6,0}(E)) \cong H^*(X_{6,0}(E)) \oplus H^*(L(2,0)) \vee L(1,0))$. Hence we see also

$$H^*(X_{6,0}(E)) \cong \mathbb{CA}\{Y\} \ominus \mathbb{CB}\{Y\} \cong \mathbb{DA}\{VY\}.$$

(See also the example after Corollary 6.8.)

Example. We consider the case $p = 7$ and $q = 2$. The cohomology $H^*(Y_{j,2}) \cong \mathbb{CA}\{y_1^j v^2, w_{-j} v^{j+2}\}$. Therefore for $r' > 4$, we see

$$H^*(X_{0,2}(G(r',e))) \cong H^*(\vee_{j=0,2,4} Y_{j,2}) \cong \mathbb{CA}\{v^2, y_1^2 v^2, y_1^4 v^2, w_4 v^4, w_2\}$$

$$H^*(X_{0,2}(G(4,e))) \cong H^*(\vee_{j=0,2,4} Y_{j,2}) \ominus H^*(L(2,2))$$
$$\cong \mathbb{CA}\{v^2, y_1^4 v^2, w_2, w_4 v^4\} \oplus \mathbb{DA}\{y_1^2 v^2\}$$

Next, we study split metacyclic groups, For stable spaces $X = X_{i,j}(G)$, let SX be the virtual object defined by

$$H^*(SX) = H^*(X) \cap \mathbb{CA} \otimes (\oplus_{q=0}^{p-2} k\{1, y_1, ..., y_1^{p-2}\}\{v^q\})$$

where we identify it as the submodule of $\mathbb{CA} \otimes (\oplus_q S(A)^*\{v^q\}) \subset H^*(E)$ in Theorem 6.1. Then we see

$$H^*(S(BE)) \cong \mathbb{CA} \otimes (\oplus_q (k\{1, y_1, ..., y^{p-2}\}\{v^q\}) \cong k[y, v]$$

identifying $C = Y = y_1^{p-1}$ as graded modules.

Recall that $H^*(M(\ell, m, n)) \cong k[y, v]$ with $|v| = 2p^{m-\ell}$. Let us write $M = (m-1, m, n))$ so that

$$H^*(Y_q(M)) \cong \oplus_{j=0}^{p-2} H^*(SY_{j,q}(E)).$$

The results in §5 imply the following theorem

Theorem 8.3. *Let $M = M(m-1, m, n)$ and $r > 3$. Then we have*

$$H^*(X_q(M)) \cong \begin{cases} \oplus_j^{p-2} H^*(SX_{j,q}(C(r))) & if\ (m, n) \neq (2, 1) \\ \oplus_j^{p-2} H^*(SX_{j,q}(E)) & if\ (m, n) = (2, 1). \end{cases}$$

Proof. The case $q \neq 0$ is shown from

$$H^*(Y_q(M)) \cong H^*(X_q(M)) \cong H^*(X_q(M(1, 2, 1)) \vee L(2, q)).$$

The case $q = 0$ is also proved similarly. □

At last in this section, we consider the cases $m - \ell > 1$. From the results in §5, it is almost immediate

Proposition 8.4. *Let $m - \ell > 1$. Then we have*

$$H^*(X_i(M(\ell, m, n)) \cong H^*(X_i(M(m-1, m, n)) \cap k[y, v^{p^{m-\ell-1}}].$$

From these results, we get

Theorem 8.5. *For $p \geq 5$, let P be a non-abelian p-group of $rank_p P = 2$, and $X_i(P)$ be an irreducible component of BP. Then there are graded submodules $H^*(P, j) \subset H^*(X_j(p_+^{1+2}))$ such that*

$$H^*(X_i(P)) \cong \oplus_{j \in J(i,P)} H^*(P, j)$$

for some index set $J(i, P)$. When P is not a metacyclic group, we can take $H^*(P, j) = H^*(X_j(p_+^{1+2}))$.

Example. Let $p = 7, q = 2$. Then we have

$$\oplus_{j=1}^5 H^*(SY_{j,2}(E)) \cong \mathbb{CA}\{1, y, ..., y^5\}\{v^2\}.$$

Hence we have

$$X_2(M(1, 2, 1)) \cong \oplus_j^5 H^*(SY_{i,2}(E)) \ominus H^*(S(L(2, 2) \vee L(2, 4)))$$

$$\cong \mathbb{CA}\{1, y_1, y_1^3, y_1^4, y_1^5\}\{v^2\} \oplus \mathbb{DA}\{y_1^2 v^2\}$$

which is still given in Theorem 5.3 (letting $S(L(2,2) \vee L(2,4)) = L(2,2)$).

9. NILPOTENT ELEMENTS

Let us write $H^{even}(X;\mathbb{Z})/p$ by simply $H^{ev}(X)$ so that

$$H^{ev}(G) = H^*(G) \oplus N(G)$$

where $N(G)$ is the nilpotent ideal in $H^{ev}(G)$.

At first, we consider metacyclic groups. Since BP is irreducible in non split cases, we only consider in split cases. Recall

$$P = M(\ell, m, n) = \langle a, b | a^{p^m} = b^{p^n} = 1, [a, b] = a^{p^\ell} \rangle$$

for $m > \ell \geq max(m-n, 1)$.

(I) Split metacyclic groups with $\ell > m-n$.

By Diethelm [Dim], its mod p-cohomology is

$$H^*(P;\mathbb{Z}/p) \cong k[y, u] \otimes \Lambda(x, z) \quad |y| = |u| = 2,\ |x| = |z| = 1.$$

Of course all elements in $H^*(P;\mathbb{Z})$ are (higher) p-torsion. The additive structure of $H^*(P;\mathbb{Z})/p$ is decided by that of $H^*(P;\mathbb{Z}/p)$ by the universal coefficient theorem. Hence we have additively (but not as rings)

$$H^*(P;\mathbb{Z})/p \cong H^*(\mathbb{Z}/p \times \mathbb{Z}/p;\mathbb{Z})$$
$$\cong k[y, u]\{1, \beta(xz) = yz - ux\} \qquad (1).$$

The element $u \in H^2(P;\mathbb{Z}/p)$ is reduced [Dim] from the spectral sequence

$$E_2^{*,*'} \cong H^*(P/\langle a\rangle; H^{*'}(\langle a\rangle;\mathbb{Z}/p)) \Longrightarrow H^*(P;\mathbb{Z}/p).$$

In fact $u = [u'] \in E_\infty^{0,2}$ identifying $H^2(\langle a\rangle;\mathbb{Z}/p) \cong k\{u'\}$. Hence $u|_{\langle a\rangle} = u'$. On the other hand, for the element $v = c_{p^{m-\ell}}(\eta)$ defined in §5, $v|_{\langle a\rangle} = (u')^{p^{m-\ell}}$ because the total Chern class in $H^*(P;\mathbb{Z}/p)$ is

$$\sum c_i(\eta)|_{\langle a\rangle} = (1+u')^{p^{m-\ell}} = 1 + (u')^{p^{m-\ell}}.$$

Therefore we see $v = u^{p^{m-\ell}}\ mod(y, xz)$ in $H^*(P;\mathbb{Z}/p)$.

Since $H^*(P)$ is multiplicatively generated by y and v with $|v| \geq 2p$ from Theorem 5.1, the element u is not integral class (i.e. $u \notin Im(\rho)$ for $\rho : H^*(P;\mathbb{Z}) \to H^*(P;\mathbb{Z}/p)$). Therefore xz is an integral class since $dim H^2(P;\mathbb{Z})/p = 2$ from (1), and $H^2(P;\mathbb{Z}/p) \cong k\{y, u, xz\}$. Moreover we have

Lemma 9.1. *The ring of the integral classes in $H^*(P;\mathbb{Z}/p)$ is given as*

$$H^{ev}(P) \cong k[y,v]\{1, xz, xzu, ..., xzu^{p^{m-\ell}-2}\} \subset H^*(P;\mathbb{Z}/p).$$

Proof. Each element y^iu^j is not nilpotent, since $H^*(P;\mathbb{Z}/p) \cong k[y,u]$. Hence for $1 \leq j < p^{m-\ell}$, each element y^iu^j is not integral. Let $A = k[y,v]\{1, xz, ..., xzu^{p^{m-\ell}-2}\}$. Then note that

$$A \cong H^*(P;\mathbb{Z}/p)/(\mathbb{Z}/p\{y^iu^j\}|1 \leq j < p^{m-\ell}).$$

Hence $H^{ev}(P) \subset A$ in the lemma.

On the other hand $dimA = n+1$ when the degree is $2n < 2p^{m-\ell}$ which is equal to $dimH^{ev}(P)$. □

Let us write

$$c_1 = xz,\ c_2 = xzu,\ ...,\ c_{p^{m-\ell}-1} = xzu^{p^{m-\ell}-2}.$$

Then $c_ic_j = (xz)^2u^{i+j-2} = 0$. Here recall $H^{even}(P;\mathbb{Z})$ is multiplicatively generated by $y, c_i(\eta)$ by the argument just before Theorem 5.1 (Thomas, Huebuschmann [Th],[Hu]). Hence we know

Lemma 9.2. *We have $c_i = \lambda_i c_i(\eta)\ mod(y, c_1, ..., c_{i-1})$ with $\lambda_i \neq 0 \in \mathbb{Z}/p$.*

Proof. By induction, assume the equation for $i-1$. Since $c_i = xzu^{i-1}$, it is not represented by the polynomial of $y, c_1, ..., c_{i-1}$. So it must be represented by $c_i(\eta)$ by the result of Thomas and Huebuschmann. □

Thus we get

Theorem 9.3. *Let P be a split metacyclic group $M(\ell, m, n)$ with $\ell > m-n$. Then we have*

$$H^{ev}(P) \cong k[y,v]\{1, c_1, ..., c_{p^{m-\ell}-1}\} \quad \text{with } c_ic_j = 0,$$

that is $N(P) \cong k[y,v]\{c_1, ..., c_{p^{m-\ell}-1}\}$.

As $Out(P)$ modules, $k\{c_i\} = k\{xzu^i\} \cong S_j$ when $i = j\ mod(p-1)$. Therefore we have

Corollary 9.4. *Let P be a split metacyclic group $M(\ell, m, n)$ with $\ell > m-n$. Then*

$$H^{ev}(X_i) \cong H^*(X_i) \oplus k[y,V]\{v^rc_s|r+s = i\ mod(p-1)\}$$

where $1 \leq s \leq p^{m-\ell}-1$.

(II) Split metacyclic groups $P = M(\ell, m, n)$ with $\ell = m - n$.
By also Diethelm [Dim], its mod p-cohomology is

$$H^*(P; \mathbb{Z}/p) \cong k[y, v'] \otimes \Lambda(a_1, ..., a_{p-1}, b, w)/(a_i a_j = a_i y = a_i w = 0)$$

where $|a_i| = 2i - 1, |b| = 1, |y| = 2, |w| = 2p - 1, |v'| = 2p$. So we see

$$H^*(P; \mathbb{Z}/p)/\sqrt{0} \cong k[y, v'].$$

Note that additively $H^*(P; \mathbb{Z})/p \cong H^*(p_-^{1+2}; \mathbb{Z})/p$, which is well known. In particular, we get additively

$$H^{ev}(P) \cong (k[y] \oplus k\{c_1, ..., c_{p-1}\}) \otimes k[v'] \quad (with\ c_i = a_i b)$$
$$\cong (k[y] \oplus k\{c_1, ..., c_{p-1}\}) \otimes k[v]\{1, v', ..., (v')^{p^{m-\ell-1}-1}\}.$$

Therefore $H^{ev}(P)$ is additively isomorphic to

$$H^{ev}(P) \cong \oplus_{i,j} k[v]\{a_i b(v')^j\} \oplus \oplus_j k[v, y]\{(v')^j\}$$

where $1 \le i \le p-1$ and $0 \le j \le p^{m-\ell-1} - 1$. Here $a_i b(v')^j$ is nilpotent and hence integral class and let $c_{jp+i} = a_i b(v')^j$. The element (v') is not nilpotent and we can take as the integral class wb of dimension $2p$. Let us write $c_{pj} = wb(v')^{j-1}$. Thus we have

Theorem 9.5. *Let P be a split metacyclic group $M(\ell, m, n)$ with $\ell = m - n$. Then*

$$H^{ev}(P) \cong k[y, v] \oplus k[y, v]\{c_i | i = 0\ mod(p)\} \oplus k[v]\{c_i | i \neq 0\ mod(p)\}$$

where i ranges $1 \le i \le p^{m-\ell} - 1$. Here the multiplications are given by $c_i c_j = 0$ for $0 < i, j < p^{m-\ell}$ and $yc_k = 0$ for $k \neq 0\ mod(p)$.

Hence we have

Corollary 9.6. *Let $P = M(\ell, m, n)$ for $\ell = m - n$. Then*

$$H^{ev}(X_i) \cong H^*(X_i) \oplus k[y, V]\{v^r c_s | s = 0\ mod(p),\ r + s = i\ mod(p-1)\}$$
$$\oplus k[V]\{v^r c_s | s \neq 0\ mod(p),\ r + s = i\ mod(p-1)\}.$$

(III) groups $P = C(r)$ or $G(r', e)$.
Let $P = C(r)$. Then it is known ([Ya1])

$$N(P) = \begin{cases} k[v]\{c_2, ..., c_{p-2}\} & r = 3, \quad |c_i| = 2i \\ k[v]\{c_1, ..., c_{p-2}\} & r \ge 4 \end{cases}$$

Each c_i is defined as a Chern class and as $Out(C(r))$ modules, we see $k\{c_i\} \cong det^i$.

For $P = G(r, e)$, each c_i is invariant under the action c^*. Hence we have

$$N(G(r+1, e)) \cong N(C(r)).$$

Theorem 9.7. *Let $P = C(r)$ or $G(r+1,e)$ for $r \geq 3$. Then*

$$H^{ev}(X_{j,i}) \cong \begin{cases} H^*(X_{0,i}) \oplus k[V]\{v^r c_s | r+s = i \ mod(p-1)\} & j = 0 \\ H^*(X_{j,i}) \quad j \neq 0. \end{cases}$$

$$\textit{where } s \textit{ ranges} \begin{cases} 2 \leq s \leq p-2 & for\ r = 3, \\ 1 \leq s \leq p-2 & for\ r \geq 4. \end{cases}$$

10. Chow rings and motives

For a smooth quasi projective algebraic variety X over $\mathbb{C}$, let $CH^*(X)$ be the Chow ring generated by algebraic cycles of codimension $*$ modulo rational equivalence. There is a natural (cycle) map

$$cl : CH^*(X) \to H^{2*}(X(\mathbb{C}); \mathbb{Z}).$$

where $X(\mathbb{C})$ is the complex manifold of $\mathbb{C}$-rational points of X.

Let V_n be a G-$\mathbb{C}$-vector space such that G acts freely on $V_n - S_n$, with $codim_{V_n} S_n = n$. Then it is known that $(V_n - S_n)/G$ is a smooth quasi-projective algebraic variety. Then it is known that $CH^*((V_n - S_n)/G)$ is independent of the choice of V_n, S_n. Hence Totaro defines the Chow ring of BG ([To1]) by

$$CH^*(BG) = lim_{n \to \infty} CH^*((V_n - S_n)/G).$$

Moreover we can approximate $\mathbb{P}^\infty \times BG$ by smooth projective varieties from Godeaux-Serre arguments ([To1]).

Let P be a $p-group$. By the Segal conjecture, the p-complete automorphism $\{BP, BP\}$ of stable homotopy groups is isomorphic to $A(P,P)_{\mathbb{Z}_p}$, which is generated by transfers and map induced from homomorphisms. Since $CH^*(BP)$ also has the transfer map, we see $CH^*(BP)$ is an $A(P,P)$-module. For an $A(P,P)$-simple module S, recall e_S is the corresponding idempotent element and $X_S = e_S BP$ the irreducible stable homotopy summand. Let us define

$$CH^*(X_S) = e_S CH^*(BP)$$

so that the following diagram commutes

$$\begin{array}{ccc} CH^*(BP)_{(p)} & \xrightarrow{cl} & H^{2*}(BP; \mathbb{Z}_{(p)}) \\ \downarrow & & \downarrow \\ CH^*(X_S)_{(p)} & \xrightarrow{cl} & H^{2*}(X_S; \mathbb{Z}_{(p)}). \end{array}$$

For smooth schemes $X.Y$ over a field K, let $Cor(X,Y)$ be the group of finite correspondences from X to Y (which is a $\mathbb{Z}_p$-module on the set of closed subvarieties of $X \times_K Y$ which are finite and surjective

over some connected component of X. Let $Cor(K, \mathbb{Z}_p)$ be the category of smooth schemes whose groups of morphisms $Hom(X, Y) = Cor(X, Y)$.Voevodsky constructs the triangulated category $DM = DM(K, \mathbb{Z}_p)$ which contains the category $Cor(K, \mathbb{Z}_p)$ (and *limit* of objects in $Cor(K, \mathbb{Z}_p)$).

Lemma 10.1. *Let S be a simple $A(P, P)$-module. Then there is a motive $M_S \in DM(\mathbb{C}, \mathbb{Z}_p)$ such that*

$$CH^*(M_S) \cong CH^*(X_S) = e_S CH^*(BP).$$

Proof. Let P act freely on $V - S$ so that $(V - S)/G$ approximates BG. For a subgroup $i : H \subset P$, the induced map i^* is defined from the projection

$$pr : (V - S)/H \to (V - S)/G.$$

This corresponds an element in morphism in $Cor(\mathbb{C}, \mathbb{Z}_p)$

$$pr^* = \{(x, pr(x)) | x \in (V - S)/H\} \in Cor((V - S)/H, (V - S)/G).$$

The transfer map is induced from

$$tr = \{(pr(x), x) | x \in (V - S)/H\} \in Cor((V - S)/G, (V - S)/H)$$

also by the definition of (finite) correspondences. Therefore each element in $A(P, P)$ is represented by a morphism of the category $DM = DM(\mathbb{C}, \mathbb{Z}_p)$. Moreover DM is a triangulated category and $Im(e_S)$ (i.e. the cone of e_S) is an object of DM. □

Remark. Of course M_S is (in general) not irreducible, while X_S is irreducible.

The category $Chow^{eff}(K, \mathbb{Z}_p)$ of (effective) pure Chow motives is defined as follows. An object is a pair (X, p) where X is a projective smooth variety over K and p is a projector, i.e. $p \in Mor(X, X)$ with $p^2 = p$. Here a morphism $f \in Mor(X, Y)$ is defined as an element $f \in CH^{dim(Y)}(X \times Y)_{\mathbb{Z}_p}$. We say that each $M = (X, p)$ is a (pure) motive and define the Chow ring $CH^*(M) = p^* CH^*(X)$, which is a direct summand of $CH^*(X)$. We identify that the motive $M(X)$ of X means $(X, id.)$. (The category $DM(K, \mathbb{Z}_p)$ contains the category $Chow^{eff}(K, \mathbb{Z}_p)$.)

It is known that we can approximate $\mathbb{P}^\infty \times BP$ by smooth projective varieties from Godeaux-Serre arguments ([To1]). Hence we can get the following lemma since

$$CH^*(X \times \mathbb{P}^\infty) \cong CH^*(X)[y] \quad |y| = 1.$$

Lemma 10.2. *Let S be a simple $A(P,P)$-module. There are pure motives $M_S(i) \in Chow^{eff}(\mathbb{C}, \mathbb{Z}_p)$ such that*

$$lim_{i\to\infty} CH^*(M_S(i)) \cong CH^*(X_S)[y], \quad deg(y) = 1.$$

The following theorem is proved by Totaro, with the assumption $p \geq 5$ but without the assumption of transferred Euler classes (since it holds when $p \geq 5$).

Theorem 10.3. *(Theorem 14.3 in [To2]) Suppose $rank_p P \leq 2$ and P has a faithful complex representation of the form $W \oplus X$ where $dim(W) \leq p$ and X is a sum of 1-dimensional representations. Moreover $H^{ev}(P)$ is generated by transferred Euler classes. Then we have $CH^*(P)/p \cong H^{ev}(P)$.*

Proof. (See page 179-180 in [To2].) First note the cycle map is surjective, since $H^{ev}(P)$ is generated by transferred Euler classes. Using the Riemann-Roch without denominators, we can show

$$CH^*(BP)/p \cong H^{2*}(P;\mathbb{Z})/p \quad for \ * \leq p.$$

By the dimensional conditions of representations $W \oplus X$ and Theorem 12.7 in [To], we see the following map

$$CH^*(BP)/p \to \prod_V CH^*(BV) \otimes_{\mathbb{Z}/p} CH^{\leq p-1}(BC_P(V))$$

$$\to \prod_V H^*(V;\mathbb{Z}/p) \otimes_{\mathbb{Z}/p} H^{\leq 2(p-1)}(C_P(V);\mathbb{Z}/p)$$

is also injective. Here V ranges elementary abelian p-subgroups of P and $C_P(V)$ is the centralizer group of V in P. So we see that the cycle map is also injective. □

Therefore we have

Corollary 10.4. *Let P be $C(r), G(r', e)$ or split metacyclic groups with $m - \ell = 1$. Then $CH^*(BP)/p \cong H^{ev}(BG)$.*

Totaro computed $CH^*(BP)/p$ for split metacyclic groups with $m - \ell = 1$ in 13.12 in [To]. When P is the extraspecial p-groups of order p^3, the above result is proved in [Ya3].

Theorem 10.5. *Let P be a split metacyclic p-group $M(\ell, m, n)$ with $m - \ell = 1$, $C(r)$ for $p \geq 3$, or $G(r', e)$ for $p \geq 5$. Then for each simple $A(P,P)$-module S, there is a motive $M_S \in DM(\mathbb{C}, \mathbb{Z}_p)$ with*

$$CH^*(M_S)/p \cong H^{ev}(X_S) = H^{even}(X_S;\mathbb{Z})/p.$$

For a cohomology theory $h^*(-)$, define the $h^*(-)$-theory topological nilpotence degree $d_0(h^*(BG))$ to be the least nonnegative integer d such that the map

$$h^*(BG)/p \to \prod_{V:el.ab.} h^*(BV) \otimes h^{\leq d}(BC_G(V))/p$$

(where V ranges elementary abelian p-subgroups of G) is injective. Note that $d_0(H^*(BG;\mathbb{Z})) \leq d_0(H^*(BG;\mathbb{Z}/p))$.

Totarto computed it in the many cases of groups P with $rank_pP = 2$. In particular, if P is a split metacyclic p-group for $p \geq 3$, then $d_0(H^*(P;\mathbb{Z}/p)) = 2$ and $d_0(CH^*(BP)) = 1$ when $m - \ell = 1$. Hence $d_0(H^*(P;\mathbb{Z})) = 2$ for these split metacyclic groups P (for $p \geq 3$).

This fact also is shown directly from Theorem 9.3 and 9.5. Let $P = M(\ell, m, n)$ with $m - \ell = 1$. Consider the restriction map

$$H^{ev}(P) \to H^{ev}(V) \otimes H^2(P;\mathbb{Z})/p \quad (where\, V = \langle a^{p^{m-1}} \rangle \subset Z(P) : center)$$

induced the product map $V \times P \to P$. Let $\ell = m-1 > m-n$. Then the element is defined in Lemma 9.1, 9.2

$$c_j = xzu^{j-1} \mapsto \sum_i u^{j-i-1} \otimes xzu^i \equiv u^{j-1} \otimes c_1 \neq 0 \in H^{ev}(V) \otimes H^2(P;\mathbb{Z})/p.$$

For $\ell = m - n$, we can see $d_0(H^*(P;\mathbb{Z})) = 2$ similarly.

References

[Be-Fe] D. J. Benson and M. Feshbach, Stable splittings of classifying spaces of finite groups, Topology 31 (1992), 157-176.

[Ca] G. Carlsson, Equivariant stable homotopy and Segal's Burnside ring conjecture, Ann. Math. 120 (1984), 189-224.

[Dim] T. Diethelm, The mod p cohomology rings of the nonabelian split metacyclic p-groups. Arch. Math 44, (1985), 29-38. 93-103.

[Di] J. Dietz, Stable splitting of classifying space of metacyclic p-groups, p odd. J. Pure and Appied Algebra 90 (1993) 115-136.

[Di-Pr] J. Dietz and S. Priddy, The stable homotopy type of rank two p-groups, in: Homotopy theory and its applications, Contemp. Math. 188, Amer. Math. Soc., Providence, RI, (1995), 93-103.

[Gr] D. Green, On the cohomology of the sporadic simple group J_4, Math. Proc. Camb. Phil. Soc. 113 (1993), 253-266.

[Gr-Le] D. Green and I. Leary, Chern classes and extra special groups. Manuscripta Math. 88 (1995) 73-84.

[Ha-Ku] J. C. Harris and N. J. Kuhn, Stable decompositions of classifying spaces of finite abelian p-groups, Math. Proc. Camb. Phil. Soc. 103 (1988), 427-449.

[Hi-Ya1] A. Hida and N. Yagita, Representation of the double Burnside algebra and cohomology of extraspecial p-group. J. Algebra **409** (2014), 265-319.

[Hi-Ya2] A. Hida and N. Yagita, Representation of the double Burnside algebra and cohomology of extraspecial p-group II. J. Algebra **451** (2016), 461-493.

[Hu] J. Huebuschmann. Chernclasses for metacyclic groups. *Arch. Math.* **61** (1993), 124-136.

[Le1] I. J. Leary, The integral cohomology rings of some p-groups, Math. Proc. Ca. Phil. Soc. 110 (1991), 25-32.

[Le2] I. J. Leary, The mod-p cohomology rings of some p-groups, Math. Proc. Camb. Phil. Soc. 112 (1992), 63-75.

[Lw] G. Lewis, The integral cohomology rings of groups of order p^3, Trans. Amer. Math. Soc. 132 (1968), 501-529.

[Ma-Pr] J. Martino and S. Priddy, The complete stable splitting for the classifying space of a finite group, Topology 31 (1992), 143-156.

[Mi-Pr] S. Mitchell and S. Priddy, Stable splitting derived from the Steiberg module. Topology 22 (1983), 285-298.

[Ni] G. Nishida, Stable homotopy type of classifying spaces of finite groups. Algebraic and Topological theories ; to the memopry of Dr. Takehiko Miyata. (1985) 391-404.

[Qu] D. Quillen, The spectrum of an equivariant cohomology ring: I, Ann. of Math. 94 (1971), 549-572.

[Ru-Vi] A. Ruiz and A. Viruel, The classification of p-local finite groups over the extraspecial group of order p^3 and exponent p, Math. Z. 248 (2004), 45-65. .

[Te-Ya] M. Tezuka and N. Yagita, On odd prime components of cohomologies of sporadic simple groups and the rings of universal stable elements, J. Algebra 183 (1996), 483-513.

[To1] B. Totaro. The Chow ring of classifying spaces. *Proc.of Symposia in Pure Math. "Algebraic K-theory" (1997:University of Washington,Seattle)* **67** (1999), 248-281.

[To2] B. Totaro. Group cohomology and algebraic cycles. *Cambridge tracts in Math.* 204 (2014).

[Th] C.B.Thomas. Characteristic classes and 2-modular representations for some sporadic groups. *Lecture note in Math. Vol. 1474* (1990), 371-381.

[Ya1] N. Yagita. Cohomology for groups of $rank_p G = 2$ and Brown-Peterson cohomology *J. Math. Soc. Japan.* **45** (1993) 627-644.

[Ya2] N. Yagita, Stable splitting and cohomology of p-local finite groups over the extraspecial p-group of order p^3 and exponent p, Geometry and Topology Monographs 11 (2007), 399-434.

[Ya3] N. Yagita. Chow rings of nonabelian p-groups of order p^3. *J. Math. Soc. Japan.* **64** (2012) 507-531.

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